Substitution Rules for Higher-Dimensional Paperfolding Structures

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Paperfolding in d Dimensions

Paperfolding sequences are a well-known example of aperiodic sequences. They can be constructed by recursion, or by a primitive substitution.

Ben-Abraham et al. (Acta Cryst. A **69** (2013) 123–130) gave a generalisation to arbitrary dimensions, in terms of a recursion.

In this talk, we show that these d-dimensional paperfolding structures can be generated by a primitive substitution as well.

This has immediate consequences on their properties, and makes available a wealth of tools for the study of further properties.

In particular, we can show that they have pure-point spectrum.

Construction by Recursion

Starting with $S_1(0) = \emptyset$, the 1-dimensional paperfolding sequence is defined on the alphabet $A = \{+, -\}$ ($+$ for valley and $-$ for crest) by the recursion

$$S_1(n+1) = m' S_1(n) \ + \ S_1(n)$$

The operation m' reflects the sequence and swaps valleys and crests.

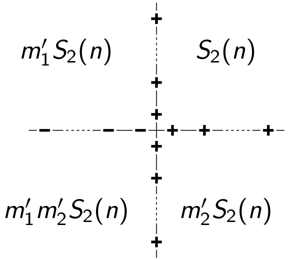
In any dimension, we always fold the negative half-space onto the positive half-space, along the coordinate axis $1, 2, \ldots, d$ (in this order).

A sequence of d consecutive such folds is called a d-fold.

Construction by Recursion: Two Dimensions

Let $S_d(n)$ be the result of n consecutive d-folds. Starting with an unfolded sheet, $S_2(0) = \emptyset$, the recursion in two dimensions then becomes

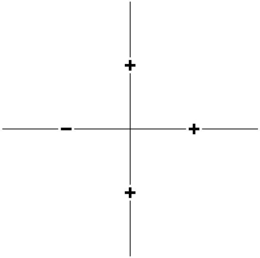
$S_2(n+1) =$



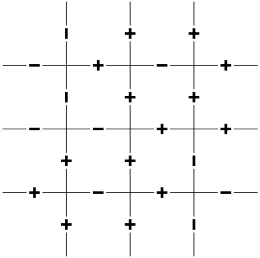
m'_a is the reflection along the x_a-axis, followed by a sign swap.

Construction by Recursion: Two Dimensions

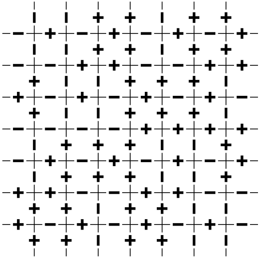
The first three iterations in two dimensions:



$S_2(1)$



$S_2(2)$



$S_2(3)$

Substitution

The recursion extends the folded sheet by appending reflected copies.

To define a substitution, we first dissect the structure into semi-cubes. These contain only half of their faces, one of each parallel pair (the one with lower coordinates), including the sign of the creases on those faces.

Under the substitution, the structure is first scaled by a factor 2, and each semi-cube is replaced locally by a 2^d block of semi-cubes.

Lemma: *Let s be a unit interval with left end point $x \in \mathbb{Z}$ in the crease pattern of $S_1(n)$, for $n \geq 2$. When folding the paper n times together, s is facing downwards in the pile if and only if x is even.*

Substitution

Consequence: The sign of the next fold in the middle of a (1d) semi-cube depends (only) on the parity of the left end point.

As we have two possible creases on the left end point, and two possible parities, we need four semi-cube types in dimension one.

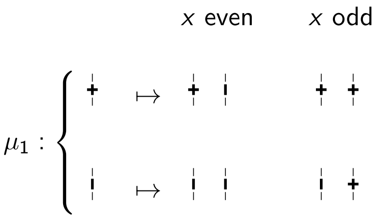
In d dimensions, it works the same, except that there is now a parity for each of the d directions; a semi-cube can have 2^d orientations on the pile.

Under substitution, depending on the parity, 2^d different d-folds are added to the interior of a semi-cube, which are all reflections of each other.

Substitution in One Dimension

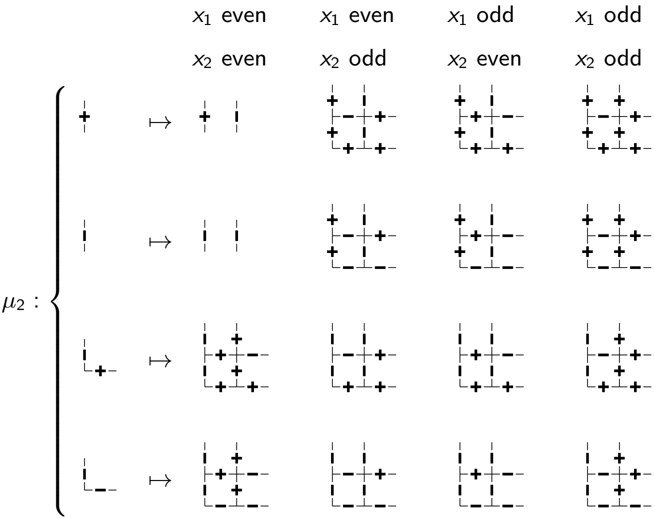
In one dimension, a semi-cube is a unit interval, including the crease sign on the left end point (the reference point), but not on the right end point.

The substitution depends on the parity of the reference point x:



μ_1 can be written as a symbolic substitution on a four letter alphabet.

Substitution in Two Dimensions



Paperfolding Substitution is Primitive

Theorem *The paperfolding substitutions μ_d are all primitive.*

We have to show that for all 2^d parity combinations, semi-cubes with all 2^d crease combinations occur.

Consider the local crease pattern at a point x. The mirror images of x have the same parity, and carry reflected copies of the local crease pattern of x.

Every local crease pattern is obtained by d folds (in some order). By the structure of a d-fold, it the set its reflections, all crease combinations occur in the first orthant ("upper right corner").

Consequence: We have a dynamical system on the associated hull, whose (translation) action is minimal and uniquely ergodic.

Spectrum and Complexity

Theorem: *The d-dimensional paperfolding structures have pure-point diffraction and dynamical spectrum.*

It is enough to show there exists a coincidence in the sense of Dekking. Indeed, if we substitute any semi-cube once, the semi-cube in the upper right corner has parities all even. Substitution once more, the semi-cube in the upper right corner is then the same for any starting semi-cube.

Theorem: *The number of distinct cubic subpatterns of linear size n grows at most as $const \cdot n^d$.*

This is a direct consequence of the substitution structure.

Cohomology of the Hull

The hull of substitution structures can be constructed as an inverse limit (Anderson and Putnam, Ergod. Th. & Dynam. Syst. **18** (1998) 509–537).

This construction allows to compute the Čech cohomology groups of the hull. Using a computer program for arbitrary block substitutions in dimensions 1 and 2, we have computed:

Theorem: *The hull of the classical 1-dimensional paperfolding structures has Čech cohomology groups* $\check{H}^0 = \mathbb{Z}$ *and* $\check{H}^1 = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$*. The hull of the 2-dimensional paperfolding structures has Čech cohomology groups*

$$\check{H}^0 = \mathbb{Z}, \quad \check{H}^1 = \mathbb{Z}[\tfrac{1}{2}]^2, \quad \check{H}^2 = \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}[\tfrac{1}{2}]^2 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}_2.$$