Response – Correlation Inequality in Dynamical Systems

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Theoretician’s Dilemmas

An exact inequality vs. an approximate equality?

Outline

Derivation of the general response-correlation inequality

Special classes: 1) Generalized Hamiltonian Systems

2) Galilean Invariant Systems

Surface growth problems:

- Theoretical implications:

  a test case for various analytical methods to dynamics

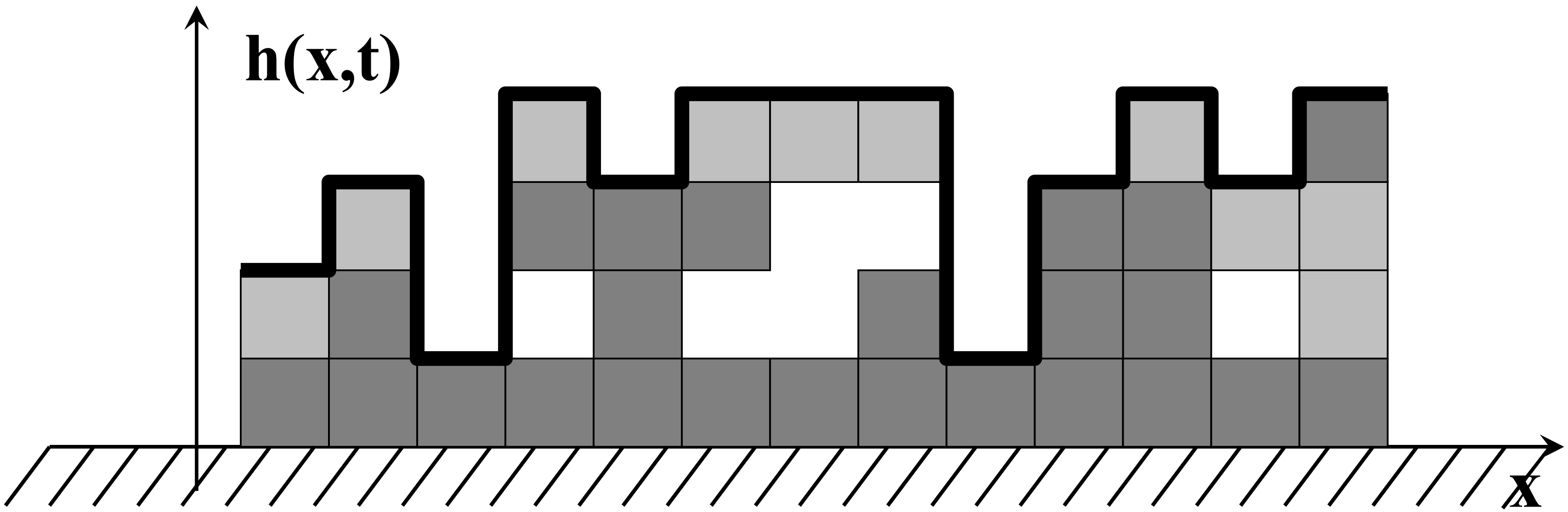
- Numerical and experimental implications

Derivation 1

Field of interest $\phi(\mathbf{q},\omega) = \phi\{\mathbf{q},\omega;\eta(\mathbf{l},\sigma)\}$ (e.g., height field)

Driven by a Gaussian noise field $\eta(\mathbf{l},\sigma)$

$$\langle\eta(\mathbf{l},\sigma)\rangle = 0 \qquad \langle\eta(\mathbf{l},\sigma)\eta(\mathbf{m},\varsigma)\rangle = 2D_0(l,\sigma)\delta(\mathbf{l}+\mathbf{m})\delta(\sigma+\varsigma)$$



Derivation 2

The Response function $\left\langle \frac{\delta\phi\left(\mathbf{q},\omega\right)}{\delta\eta\left(\mathbf{p},\sigma\right)} \right\rangle \equiv G\left(q,\omega\right)\delta\left(\mathbf{q}-\mathbf{p}\right)\delta\left(\omega-\sigma\right)$

Correlation function $\left\langle \phi\left(\mathbf{q},\omega\right)\phi\left(-\mathbf{p},-\sigma\right) \right\rangle \equiv \Phi\left(q,\omega\right)\delta\left(\mathbf{q}-\mathbf{p}\right)\delta\left(\omega-\sigma\right)$

Gaussian character of the noise gives

$$G\left(q,\omega\right)\delta\left(\mathbf{q}-\mathbf{p}\right)\delta\left(\omega-\sigma\right) = \left\langle \phi\left(\mathbf{q},\omega\right)\eta\left(-\mathbf{p},-\sigma\right) \right\rangle \big/ 2D_0\left(q,\omega\right)$$

Which can be used as a definition of the Response function even when the noise is not Gaussian

The average $\left\langle \chi\left(\mathbf{q},\omega\right)\psi\left(-\mathbf{p},-\sigma\right) \right\rangle$ can be viewed as a scalar product of $\chi\left(\mathbf{q},\omega\right)$ and $\psi\left(-\mathbf{p},-\sigma\right)$,

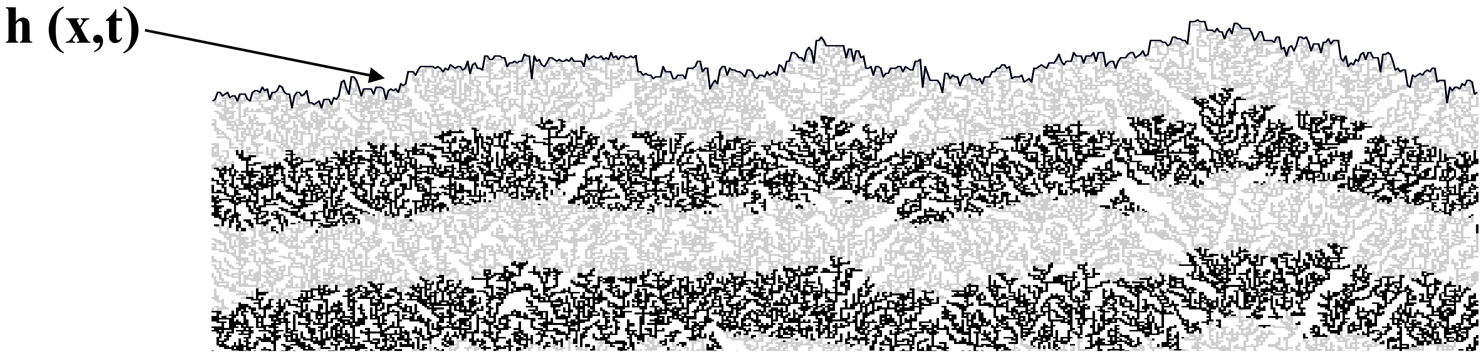
thus using the Cauchy-Schwartz inequality we obtain $\left|\left\langle \chi,\psi \right\rangle\right|^2 \le \left\langle \chi,\chi \right\rangle \left\langle \psi,\psi \right\rangle$

$$\left|G(q,\omega)\right|^2 D_0(q,\omega) \le \Phi(q,\omega)$$

Exponent inequality 1

Often the following scaling is obeyed:

The equal-time correlation function $\Lambda(q) = \int_{-\infty}^{\infty} \Phi(q,\omega)\, d\omega \propto q^{-\Gamma}$



$$\left\langle \left[h(x,t) - h(x',t) \right]^2 \right\rangle \propto |x - x'|^{2\alpha}$$

Algebraic correlations = The interface is self-affine

with a roughness exponent α $\quad (\Gamma = d + 2\alpha)$

Exponent inequality 2

Characteristic frequencies

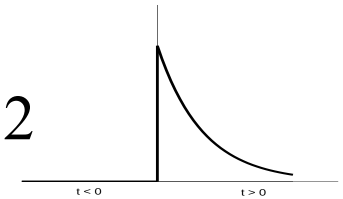
$$\omega_R^{-1}(q) = G(q, \omega = 0) \propto q^{\bar{z}}$$

$$\omega_C^{-1}(q) = \pi\Phi(q, t = 0)/\Lambda(q) \propto q^{z}$$

An example – the Edwards-Wilkinson model $\partial_t h(\mathbf{r}, t) = \nu\nabla^2 h + \eta(\mathbf{r}, t)$

$$G(q, \omega) = \frac{1}{\nu q^2 - i\omega} \quad \Rightarrow \quad \omega_R^{-1}(q) = \nu q^2 \quad \Rightarrow \quad \bar{z} = 2$$

$$\Phi(q, t) = \frac{D_0}{\nu q^2} \exp\left[-\nu q^2 t\right] \quad \Rightarrow \quad \omega_C^{-1}(q) = \nu q^2 \quad \Rightarrow \quad z = 2$$



So that $z = \bar{z}$, and this is the case for linear systems, for example.

Actually also $\Gamma = z$

Exponent inequality 3

We consider a noise correlator (spectral function) of the following form

$$D_0\left(q,\omega\right) = Bq^{-2\sigma} \quad \text{with} \quad \sigma \geq 0 \quad (\text{and } \sigma = 0 \text{ is white noise})$$

Setting $\omega = 0$ in the general ineqaulity we obtain the exponent ineqaulity

$$\left|G(q,\omega)\right|^2 D_0(q,\omega) \leq \Phi(q,\omega)$$

We obtain:

$$2\overline{z} + 2\sigma \leq \Gamma + z$$

Special cases: Hamiltonian systems

Whenever a Fluctuation-Dissipation Relation exists – such as for Hamiltonian systems, $G \propto \Phi$ then we get $\bar{z} = \Gamma - 2\sigma$

see Hohenberg & Halperin, **Theory of dynamic critical phenomena, Rev Mod Phys 49,** 435 (1977). $\partial_t \varphi(\mathbf{r},t) = -\frac{\delta H}{\delta \varphi} + \eta(\mathbf{r},t) \quad (\text{white noise})$

So whenever $\bar{z} = \Gamma - 2\sigma$: $\quad z \geq \Gamma - 2\sigma$

e.g., for φ^4 (Model A): $\quad \partial_t \varphi(\mathbf{r},t) = D\nabla^2\varphi + \mu\varphi - g\varphi^3 + \eta(\mathbf{r},t)$

with white noise (i.e., $\sigma = 0$) we obtain: $\quad z \geq \Gamma$

Which is indeed satisfied by the RG result (HH 1977): $\begin{cases} \Gamma = 2 - \eta \\ z = 2 + c\eta \end{cases} + O\left((4-d)^2\right)$

(Perturbatively though…) and for d < 4, z > Γ

Special cases: Hamiltonian Systems

For φ^4 (Model B): $\partial_t \varphi(\mathbf{r},t) = \nabla^2 \left[D\nabla^2 \varphi + \mu\varphi - g\varphi^3 \right] + \eta(\mathbf{r},t)$

with conservative noise (i.e., σ = -1) we obtain again: $z \geq \Gamma + 2$

Which is indeed satisfied by the RG result (HH 1977): $\begin{cases} \Gamma = 2 - \eta \\ z = 4 - \eta \end{cases} + O\left((4-d)^2\right)$

by saturating the inequality, i.e. z = $\Gamma + 2$

Special cases: Galilean Invariance

Whenever a Galilean invariance (i.e., the equation of motion is independent on trans $h \to h + vt$) exists we get $\bar{z} = z$

For example, any proper equation in Fluid dynamics, such as ~ Navier-Stokes, Burgers' equation and also Kardar-Parisi-Zhang, Molecular Beam Epitaxy eq. Proof based on scaling.

$$\partial_t h(\mathbf{r}, t) = \nu \nabla^2 h + g(\nabla h)^2 + \eta(\mathbf{r}, t)$$

The evolution of the zero-mode h_0 is decoupled from the other modes and so $G(k=0,\omega) = 1/i\omega$ and this implies under scaling $G(k,\omega) = \frac{1}{q^{\bar{z}}} f\left(\frac{\omega}{\omega_q}\right)$ that $\underline{\bar{z} = z}$

So whenever $z = \bar{z}$: $\quad z \le \Gamma - 2\sigma \le \Gamma$

Can be quite useful …

Special cases: Summary

In general, there are three exponents $\bar{z}, z, \Gamma \quad \rightarrow \quad 2\bar{z} + 2\sigma \leq \Gamma + z$

The special cases we identified so far are:

1. A linear system: $\bar{z} = z = \Gamma$
2. Hamiltonian system: $\bar{z} = \Gamma - 2\sigma \quad \rightarrow \quad z \geq \Gamma - 2\sigma$
3. Galilean invariance: $\bar{z} = z \quad \rightarrow \quad z \leq \Gamma - 2\sigma$

Open questions:

Is there an *interesting* system where all exponents are different?
(it must be nonlinear, non-Hamiltonian and without Galilean invariance)

Is there a system with $z = \Gamma$, and different from $\bar{z}$?

Theoretical Implications

The inequality turns out to be very useful to test various theoretical approaches to dynamic critical phenomena: Dynamical RG, Mode-Coupling, Scaling Approach and the Self-Consistent Expansion

NKPZ (Nonlocal Kardar-Parisi-Zhang) equation (Mukherji & Bhattacharjee 97)

$$\partial_t h(\mathbf{r},t) = \nu\nabla^2 h + \int dr' g(\mathbf{r}')\nabla h(\mathbf{r}+\mathbf{r}',t)\cdot\nabla h(\mathbf{r}-\mathbf{r}',t) + \eta(\mathbf{r},t)$$

The kernel **g(r)** has a short-range part $\sim g_0\delta^d(\vec{r})$

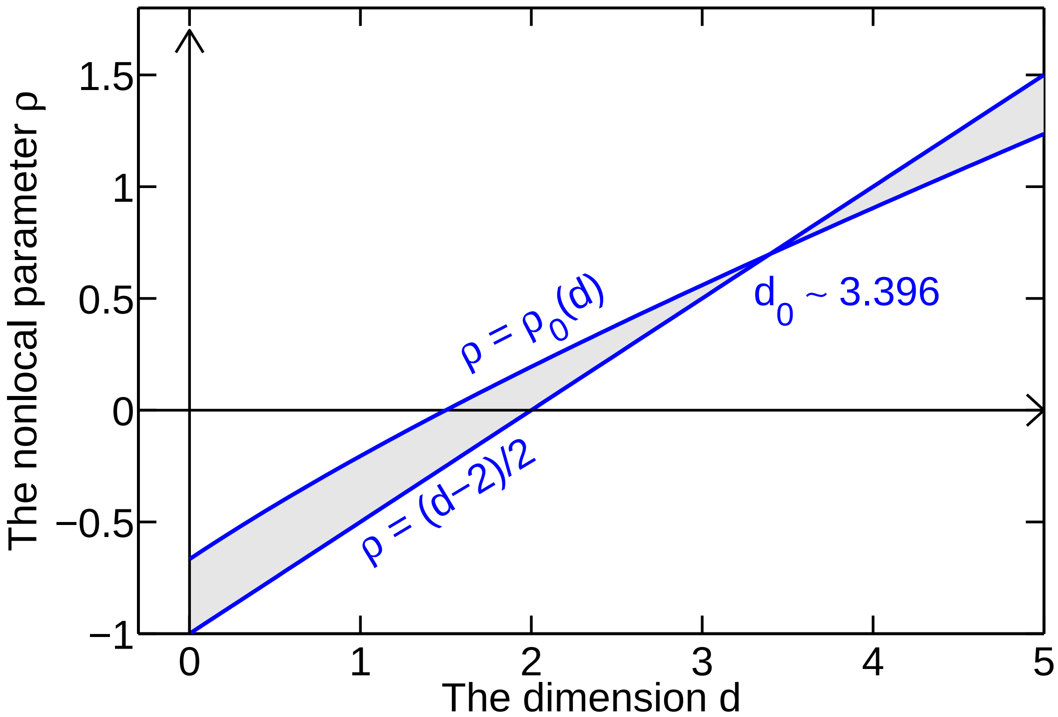
and a long-range part $\square\; g_\rho r^{\rho-d}$, or in Fourier space $\hat{g}(q) = g_\rho q^{-\rho}$

Scaling relation $\alpha + z = 2 - \rho$ (1 independent exp.)

Exact result (Katzav 02) for $d = 1, \rho = 2\sigma$: $z_{exact} = \dfrac{3-3\rho}{2}$

Results - DRG

$$z \le \Gamma - 2\sigma \le \Gamma$$



$$z_{DRG} = 2 + \frac{(d - 2 - 2\rho)(d - 2 - 3\rho)}{(3 + 2^{-\rho})d - 6 - 9\rho}$$

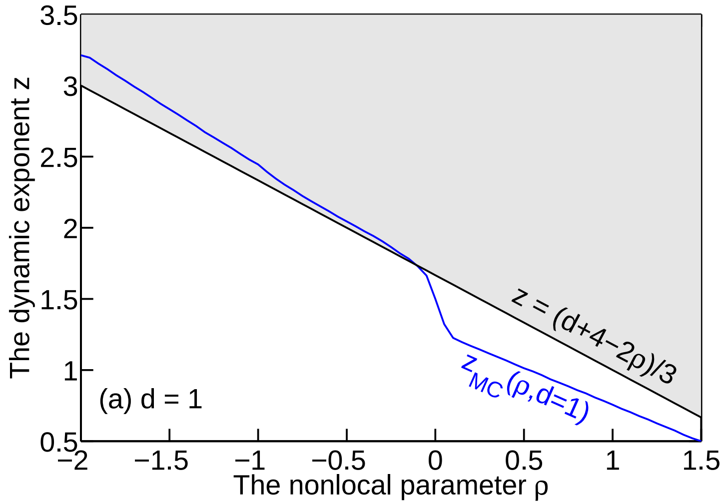
$(\sigma = 0$ - white noise$)$

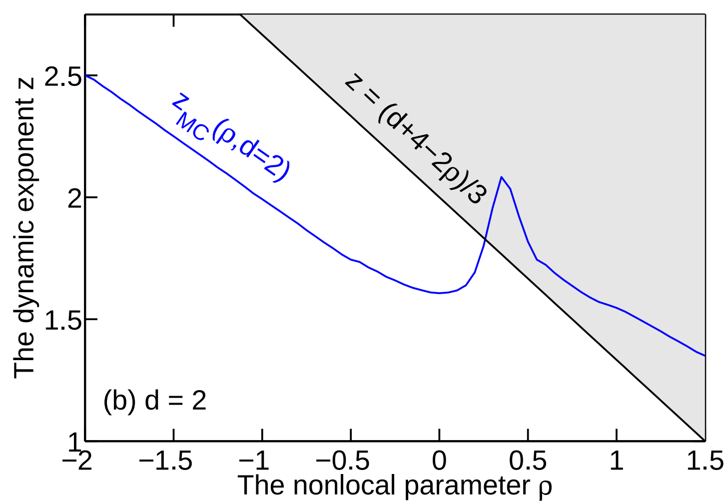
Dynamic Renormalization Group

(Mukherji & Bhattacharjee 97)

Results - MC

$$z \le \Gamma - 2\sigma \le \Gamma$$

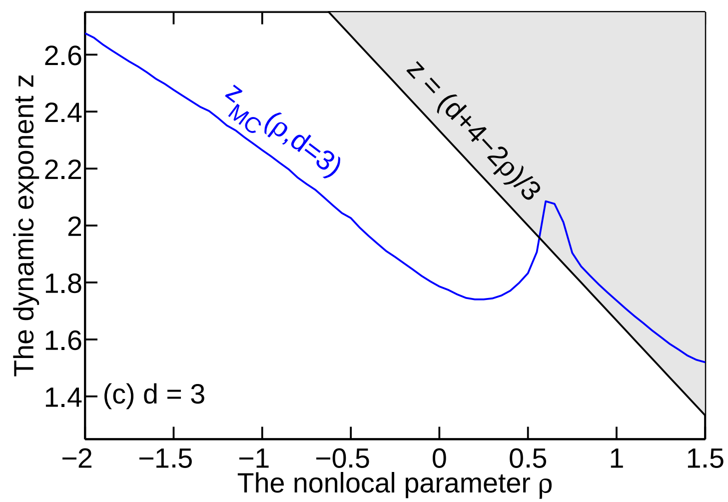




Mode Coupling

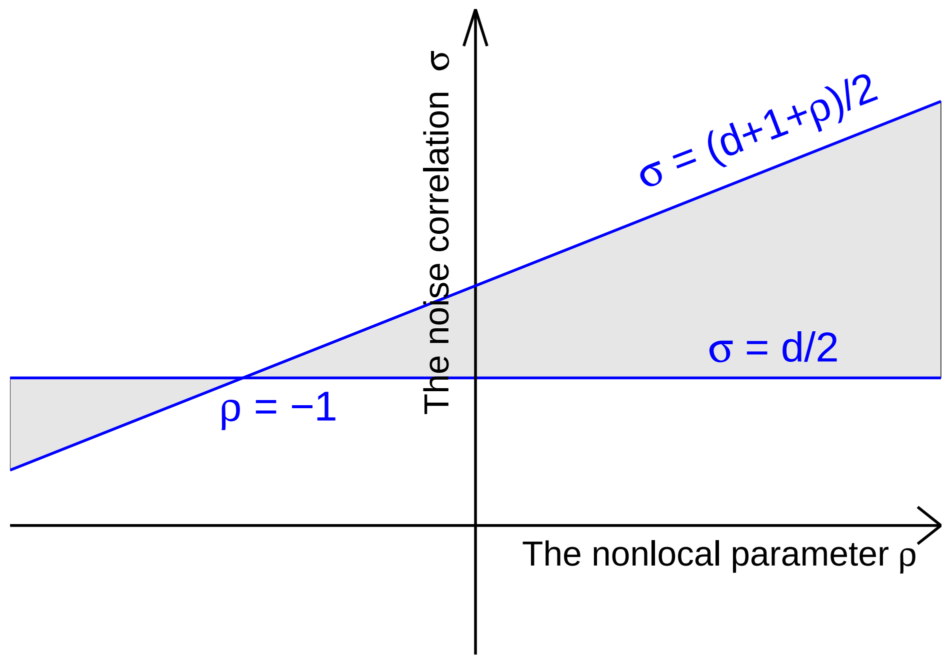
(Hu & Tang 02)

$(\sigma = 0$ - white noise$)$



Results - SA

$$z \le \Gamma - 2\sigma \le \Gamma$$



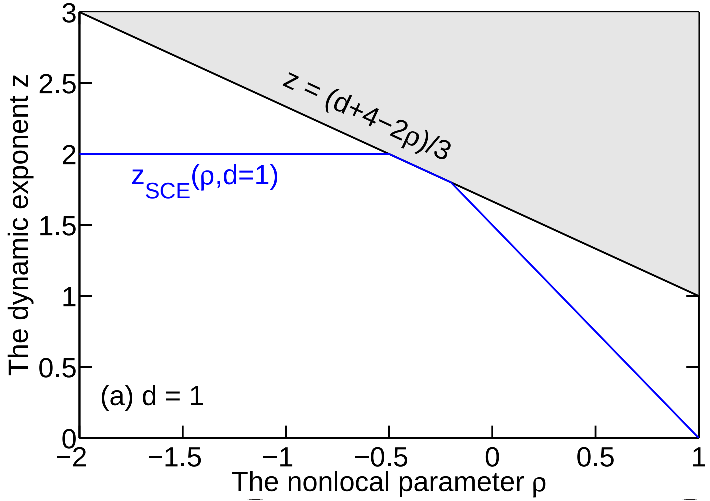
$$z_{SA} = \frac{(2-\rho)(d+2-2\sigma)}{d+3-2\sigma}$$

Scaling Approach

(Tang & Ma 97)

Results - SCE

$$z \le \Gamma - 2\sigma \le \Gamma$$



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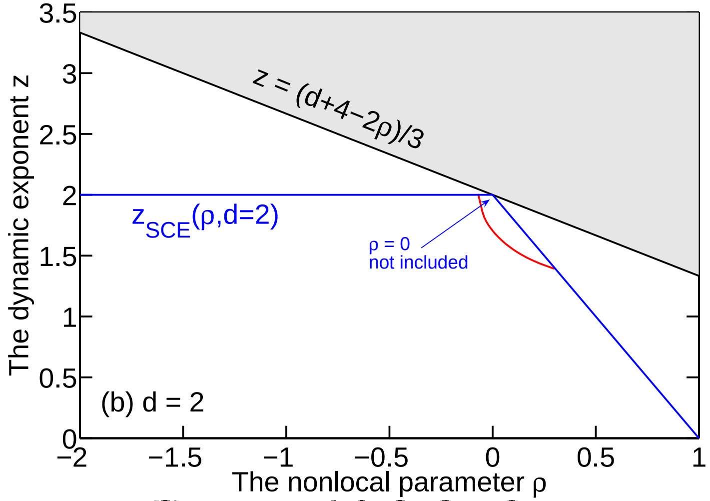
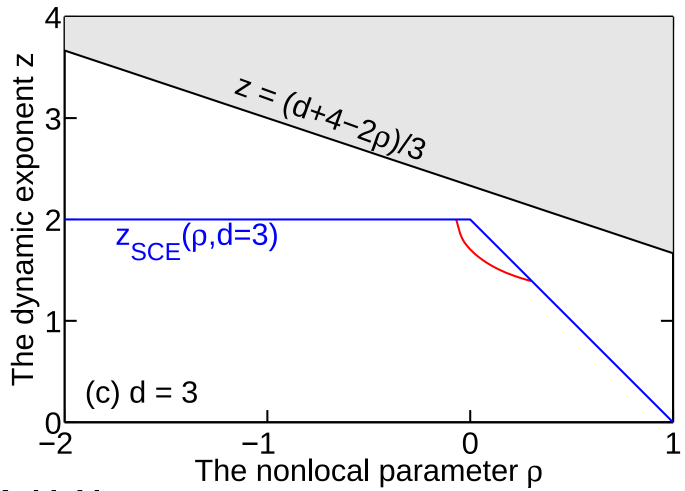


(2) $\sigma > \frac{1}{2}\rho$ and $d > 2 + 2\sigma + 2\rho$

1

$\frac{d+4-2\rho-2\sigma}{3}$ $\qquad$ $\frac{d+4-2\rho+4\sigma}{3}$

$\frac{d+4-\Gamma_0(d,\rho)-2\rho}{2}$ $\qquad$ Sol. of $F(\Gamma,z,\rho)=0$, denoted by $\Gamma_0(d,\rho)$



$(\sigma = 0$ - while noise$)$

SChwartz-Edw
Self-Consistent Ex

(Katzav & Schwartz 03)

Summary - theoretical implications

The inequality detects shortcomings of various analytical methods, and thus suggests that those methods should be used with care even in cases where the inequality is not violated until the origin of the violation is better understood

- Dynamical Renormalization Group
- Mode-Coupling
- Scaling Approach

- The SCwartz-Edwards Self-Consistent Expansion

- **Consistent with the comparison to the exact result in 1D**

Experimental Implications 1

Experimental results that violate the (exponent) inequality:

- crack surfaces in wood (Morel et al 05) and in mortar (Mourot et al 04)
- roughness in Fe-Cr superlattices (Santamaria et al 02)
- roughening in silicon nitride films by plasma enhanced chemical vapor deposition (Karabacak et al 02)
- roughening in wrinkly metal (Aurongzeb 05)
- …

The roughness (α) and dynamic (z) exponents has been measured independently and they violate the exponent inequality $z \leq \Gamma$

WHY???

Experimental Implications 2

Possible explanations:

- could be due to one or more of the numerous artifacts in the methods used to extract the exponents.

- The time involved in the measurements is too short for the system to reach steady-state, while quantities like the exponents or correlation/response function should be measured in steady-state.

Experimental Implications 3

- This can be tested by measuring experimentally $G(\mathbf{r},t)$ and $\Phi(\mathbf{r},t)$ and using the more general inequalities

$$2\overline{z} + 2\sigma \le \Gamma + z$$

$$\left|G(q,\omega)\right|^2 D_0(q,\omega) \le \Phi(q,\omega)$$

This will put the data to a much more stringent test

- **$z \ne \overline{z}$ - a very powerful constraint on the kind of theory that can describe properly this class of phenomena!!!**
  Namely, explicit dependence on the height h!

Dual Membership

- A system can be a member of both classes, i.e. derived from a Hamiltonian, and be Galilean invariant: $\bar{z} = z = \Gamma - 2\sigma$

1) $\partial_t \varphi(\mathbf{r},t) = \nabla^2 \left[D\nabla^2\varphi + \mu\varphi - g\varphi^3 \right] + \eta(\mathbf{r},t)$ where $\Gamma = 2 - \eta$, $z = 4 - \eta$

known from RG, but not generally proven.

2) The Molecular-Beam-Epitaxy equation $\partial_t h(\mathbf{r},t) = -K\nabla^4 h + \nabla^2 (\nabla h)^2 + \eta(\mathbf{r},t)$

derived from $H = \int d^d\mathbf{r} \left[\frac{K}{2} (\nabla^2 h)^2 + \frac{1}{4} (\nabla h)^4 \right]$

In this case $\Gamma = z = 4$ – from RG and SCE, but not proven.

But the scope is much wider, and model which is not easily RG-able, with a Hamiltonian which depends only on derivatives of h is in, e.g.:

3) A growing surface, minimizing surface tension: $H = \gamma \int d^d\mathbf{r} \sqrt{1 + (\nabla h)^2}$

$\partial_t h = \gamma \nabla \cdot \left(\frac{\nabla h}{\sqrt{1+(\nabla h)^2}} \right) + \eta$ must have $\Gamma = z$, reducing the exponents by one.

Summary and Outlook

- The derivation of the response-correlation inequality was based on minimal assumptions, and in no way restricted to surface growth phenomena. All one needs is -

A field of interest $\phi\left(\mathbf{q},\omega\right)=\phi\left\{\mathbf{q},\omega;\eta\left(\mathbf{l},\sigma\right)\right\}$

Driven by a (Gaussian) noise field $\eta\left(\mathbf{l},\sigma\right)$

$$\left|G(q,\omega)\right|^2 D_0(q,\omega)\leq\Phi(q,\omega)\quad\rightarrow\quad 2\overline{z}+2\sigma\leq\Gamma+z$$

So practically applicable to any dynamical physical system that is amenable to a hydrodynamic stochastic description: Turbulence, Self-Gravitating Gas, and more…

Summary and Outlook

Could be interesting to explore further implications of the inequality to various dynamical systems, and could be used as a guide to model Physical systems.

The inequality implies an equality for a class of systems.

This inequality can be generalized / extended in various ways:

1. We have found an alternative proof for the Hamiltonian class, (using a “quantum Hamiltonian” approach) lending itself to generalizations relating various correlations and responses.
2. Systems with quenched disorder.
3. Systems described by more than one field.