Dynamics of Thermalisation: a Gaussian Regime

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Introduction

How does thermalisation in closed quantum systems occur?

Main results:

- Demonstrate equilibration to thermal state in small Hubbard model subsystems coupled to a small bath
- Explore the effect of coupling between subsystem and bath on thermalisation dynamics
- Find an unexpected Gaussian form of relaxation at strong coupling strength and suggest that this is generic behaviour

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When composite system is in microcanonical state, $\rho = \omega$

$$\omega \sim \sum_s e^{-\beta\varepsilon_s}|s\rangle\langle s|$$

$$\text{with } \beta = \left(\frac{d \ln N_{\mathrm{bath}}(E)}{dE}\right)_{E_0}$$

ω is the thermal subsystem state

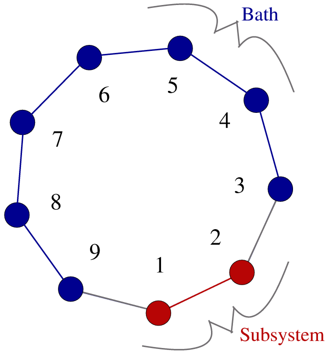
Hubbard Model

$$H_S = -\sum_{\sigma=\uparrow,\downarrow} J_\sigma (c^\dagger_{1\sigma} c_{2\sigma} + \text{h.c.}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$$

$$H_B = -\sum_{i=3}^{L-1} \sum_{\sigma=\uparrow,\downarrow} J_\sigma (c^\dagger_{i\sigma} c_{i+1,\sigma} + \text{h.c.}) + U \sum_{i=3}^{L} n_{i\uparrow}n_{i\downarrow}$$

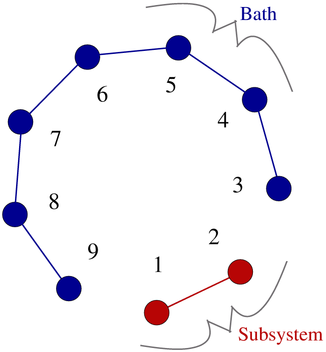
$$V = -\sum_{\sigma=\uparrow,\downarrow} J_\sigma \left[(c^\dagger_{2\sigma} c_{3\sigma} + c^\dagger_{1\sigma} c_{L\sigma}) + \text{h.c.} \right]$$

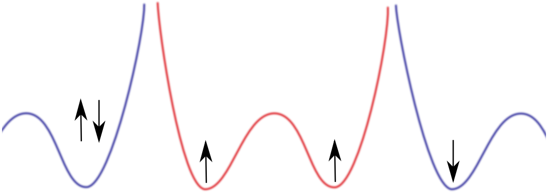
- 8 fermions: $4\uparrow$, $4\downarrow$
- $J_{ij} = J(1 + \xi \operatorname{sgn} \sigma)$, $\xi = 0.05$
- $U = J = 1$
- 15876 energy levels
- 16 subsystem energy levels
- $\lambda = 1 \rightarrow$ homogeneous ring



Initial States

- Product states
  $|\Psi(t=0)\rangle = |s\rangle \frac{1}{\sqrt{B}} \sum_{b=b_l}^{b_u} |b\rangle$
- Well-defined energy





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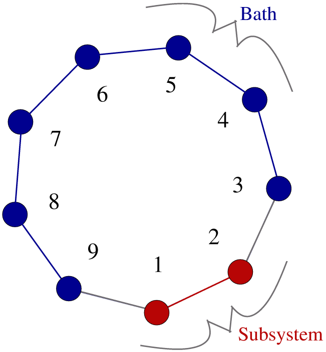
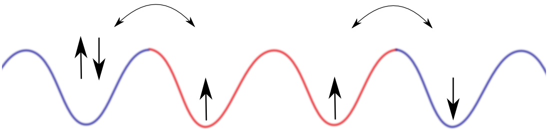
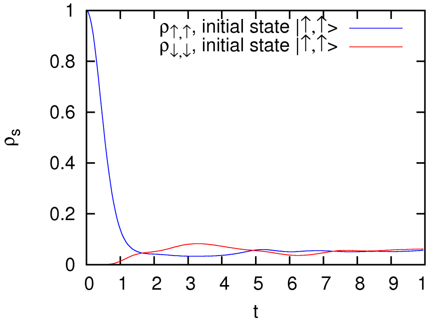
- Switch on λV for $t > 0$
- Evolve $\rho(t) = \mathrm{Tr}_{\mathrm{bath}}\left(|\Psi(t)\rangle\langle\Psi(t)|\right)$

with $|\Psi(t)\rangle = e^{-iHt}|\Psi\rangle$



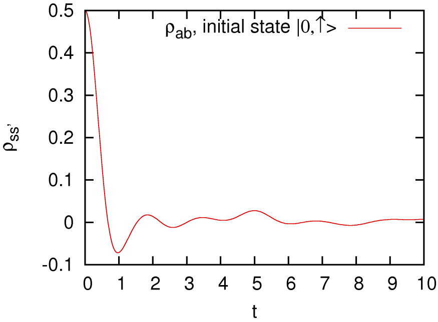


Subsystem Evolution



Diagonal elements of ρ

Steady state reached after relaxation



Off-diagonal elements of ρ

$$|s_a\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow, 0\rangle + |0, \uparrow\rangle\right)$$
$$|s_b\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow, 0\rangle - |0, \uparrow\rangle\right)$$

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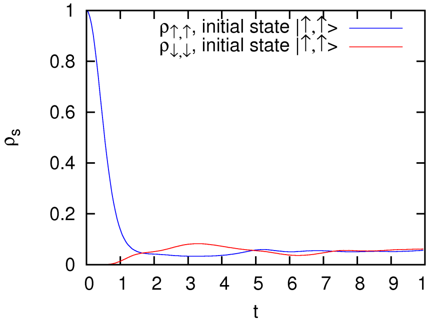

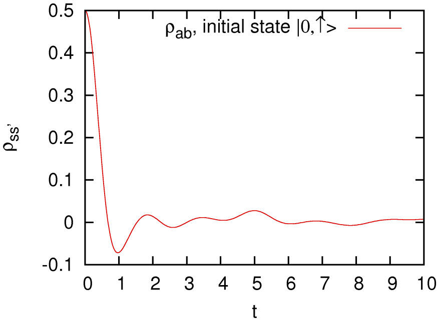

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- Steady state is the thermal state ω

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Subsystem thermalisation seen for many λ, but how does it get there?

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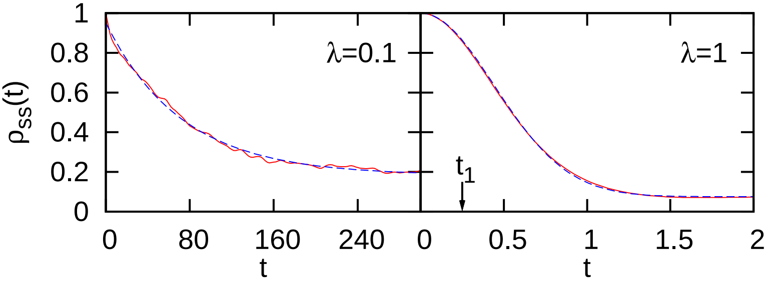
Initial state $|s\rangle = |\uparrow,\uparrow\rangle$ with composite energy $E_0 = -2$



Small $\lambda \longleftrightarrow$ Large λ

Exponential, $Ae^{-\gamma t} + \text{const} \longleftrightarrow$ Gaussian, $A'e^{-\Gamma^2 t^2} + \text{const}$

Short Time Dynamics

- "Very short" times

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle \simeq (1 - iHt)|\Psi(0)\rangle$$

$$\rho_{ss}(t) \simeq 1 - \Gamma_{\text{short}}^2 t^2 \quad \text{with} \quad \underline{\Gamma_{\text{short}} \propto \lambda}$$

- Times greater than $t_1 =$ inverse single-particle bandwidth, with small λ:
  Fermi Golden Rule behaviour:

$$\frac{d\rho_{ss}}{dt} = -\gamma_{\text{FGR}} \quad \text{with} \quad \underline{\gamma_{\text{FGR}} \propto \lambda^2}$$

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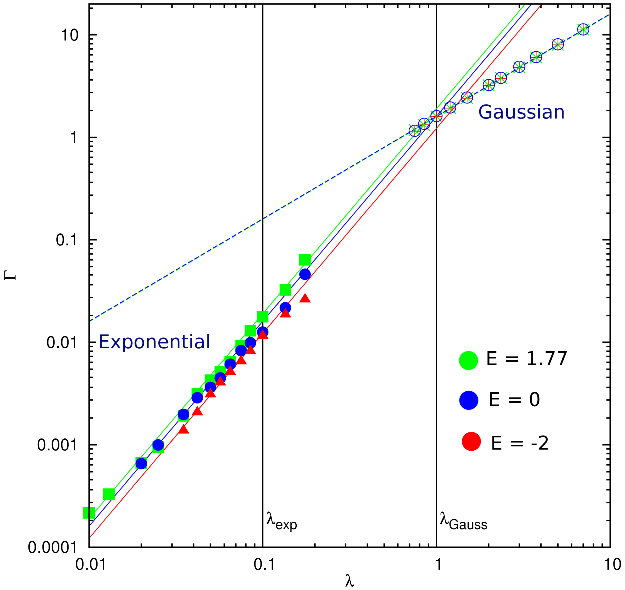
Predictive Power of Short-Time Dynamics



γ_{FGR}, Γ_{short} (lines)

Fits to Gaussian/exponential curves (points)

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  We have explored this numerically by considering:

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- We find Lorentzian to Gaussian crossover is found for $|\langle sb|E\rangle|^2$ in Brownian motion model for V with finite bandwidth for couplings

c.f. M. Wilkinson P. N. Walker, J. Phys. A: Math. Gen. **28** 6143 (1995)

Conclusions

- Small Hubbard-model systems in pure states demonstrate subsystem thermalisation for a range of coupling strengths
- Dynamics is strongly dependent on coupling strength, with Gaussian behaviour seen at moderate/strong coupling strength
- Find that Gaussian behaviour is generic and that it holds in the limit of large bath

  Genway, Ho and Lee, PRL **105** (2010) 260402
- Find generic Gaussian behaviour using a stochastic walk through subsystem-bath couplings to find $|\langle sb|E\rangle|^2$ as a function of $E - (\varepsilon_s + \epsilon_b)$