Geometrically defined extensions of the Temperley-Lieb algebras

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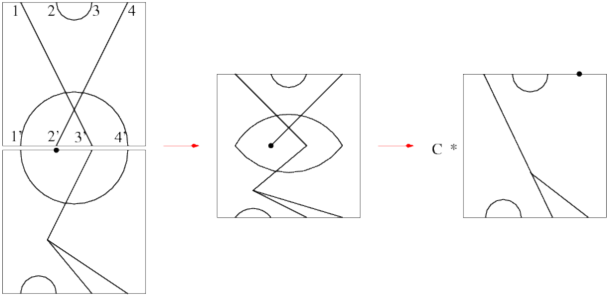
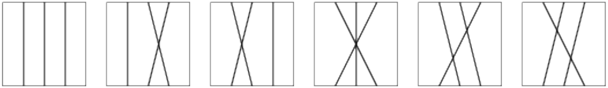
No generic definition for diagram algebras

- Basis: a finite set of diagrams [tale about categories and isotopy]
- Multiplication: concatenation, rescaling to square
- Base change: needed for closedness in most interesting cases (the field $k = \mathbb{R}, \mathbb{C}$, etc. matters)

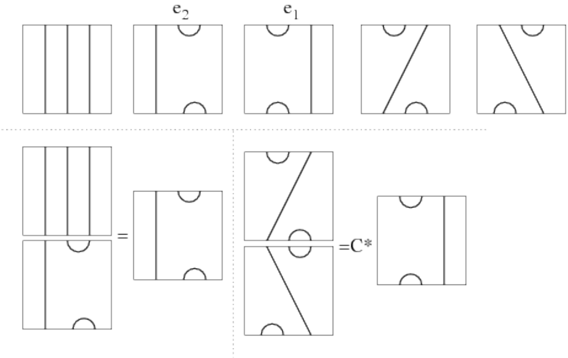


Examples

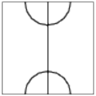
kS_n



Temperley-Lieb: $TL_n(C)$



Brauer: $B_n(C)$, basis:

$kS_n + kTL_n(C) +$ 

Definition from pictures or from relations among generators e.g., for TL_n

$$\begin{aligned} e_i e_j &= e_j e_i \text{ if } |i-j| > 1 \\ e_i e_{i\pm1} e_i &= e_i \\ e_i^2 &= C e_i \end{aligned}$$

Then diagram representation needs to be proved. E.g.,
1. Diagrams satisfy relations: algebra homomorphism.
2. Dimensions match: injective

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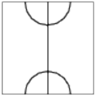
$kS_n + kTL_n(C) +$ 

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Statistical mechanics

Transfer matrices often representations of diagram algebras. E.g., $TL_n(C)$

- Q-state Potts model
- 6-vertex model
- Percolation problem

Simple (no loop, no multiple edges), undirected graphs G (edges E_G, vertices V_G), configuration: each vertex labelled by a state/spin $\{1, 2, \ldots, Q\}$

$$H(\{\sigma_i\}|_{i \in V_G}) = J \sum_{<i,j> \in E_G} \delta_{\sigma_i, \sigma_j}$$

The partition function ($J = 1$) reads

$$Z_G(\beta) = \sum_{\{\sigma_i\}} \prod_{<i,j>\in E_G} \exp(\beta\delta_{\sigma_i,\sigma_j}) = \sum_{\{\sigma_i\}} \prod_{<i,j>\in E_G} (1 + v\delta_{\sigma_i,\sigma_j})$$

with $v = \exp\beta - 1$. Further expanded

$$Z_G(\beta) = \sum_{\{\sigma_i\}} \sum_{G'\in\mathcal{P}(E_G)} \prod_{<i,j>\in E_{G'}} v\delta_{\sigma_i,\sigma_j} = \sum_{G'\in\mathcal{P}(E_G)} v^{|G'|} Q^{\#(G')}$$

If there's a boundary V (vertex set, where the configuration is fixed):

$$(Z_G^V(Q))_c = \sum_{G'\in\mathcal{P}(E_G)|G'\sim c} v^{|G'|} Q^{\#_c(G')}$$

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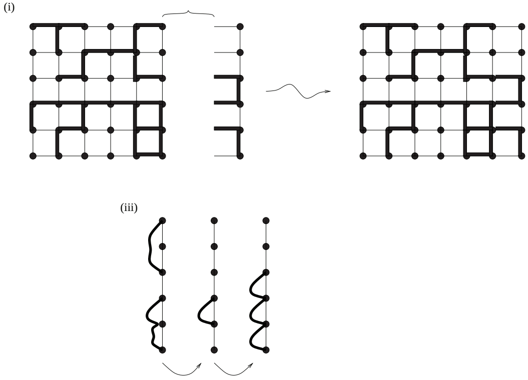

Figure 1: (i) A subgraph of a square lattice and an extra layer. (ii) The corresponding new subgraph. (iii) A sequence showing: the connectivity of the original subgraph (running $\# = 12$); the connectivity after adding the new horizontal edges (running $\# = 12+3$); the connectivity after adding the new vertical edges (running

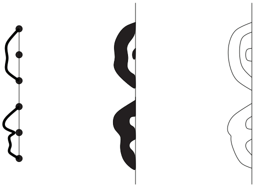

Figure 2: Mapping planar Whitney diagram to TL diagram.

The transfer matrix looks like

$$\mathcal{T} = \prod_i (1 + \frac{v}{\sqrt{Q}} R(e_{2i}) \prod_i (1 + \frac{v}{\sqrt{Q}} R(e_{2i-1})$$

R is a representation of the $TL_n(\sqrt{Q})$ induced by the construction. The main motivation (would be) for our work: Schur-Weyl duality rooted in integrable systems.
Plus quantum spin chains (losely related)

In 1401.1774 we defined $J_{l,n}$ and investigated the representation theory

- determined the index set for the simple modules
- discussed the Bratelli diagram
- gave some examples of non-semisimple cases

Beside representation theory, plenty of questions remain

- Schur-Weyl duality?
- extension to BMW? (in progress)
- applications? (spin chain, stat. mech., etc.)