Alex Clark

University of Leicester

Open Statistical Physics
Aperiodic Order Session

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Flow Equivalence

Definition

Homeomorphisms $f_i : X_i \to X_i$ of Cantor sets are *flow equivalent* if there is a homeomorphism

$$h : \mathrm{Susp}(f_1; X_1) \to \mathrm{Susp}(f_2; X_2)$$

between their suspensions that preserves the orientation of the respective flows.

In studying the flow equivalence of subshifts of finite type, Parry and Sullivan (*Topology*, 1975) show that the flow equivalence relation on homeomorphisms of Cantor sets is generated by :

- topological conjugacy and
- expansion.

Parry-Sullivan

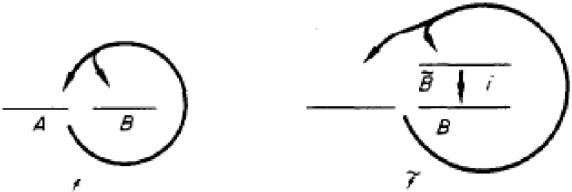


Fig. 1.

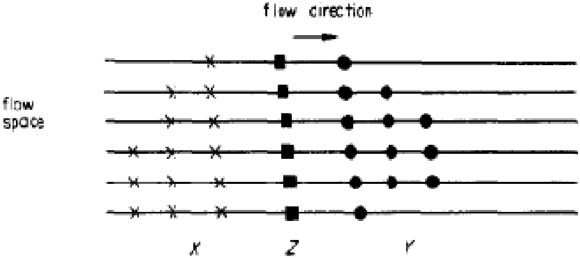


Fig. 2.

Parry-Sullivan

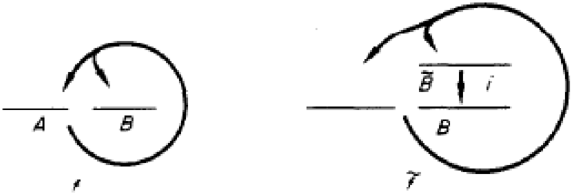


Fig. 1.

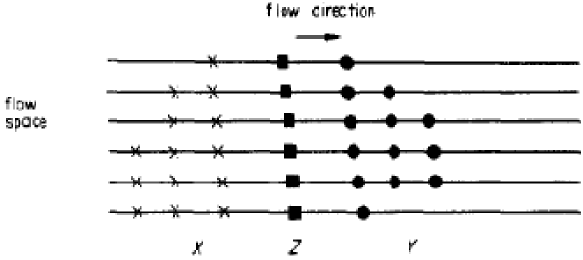


Fig. 2.

Parry-Sullivan

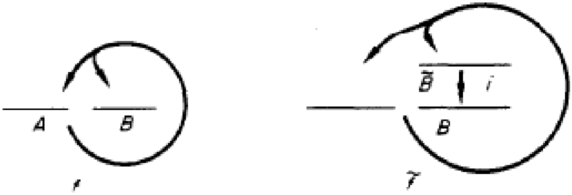


Fig. 1.

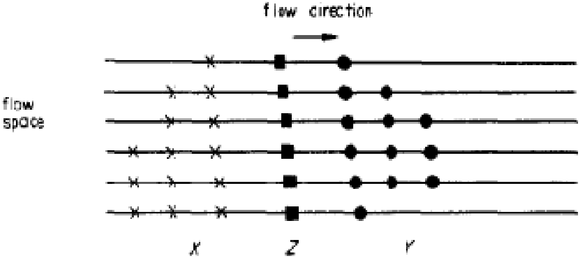
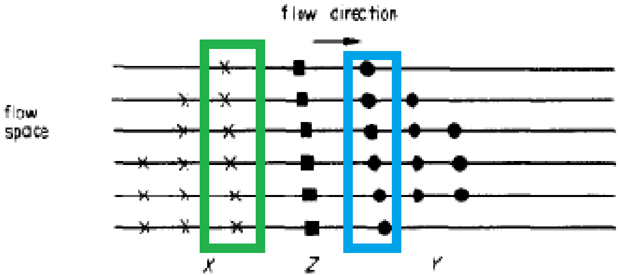


Fig. 2.

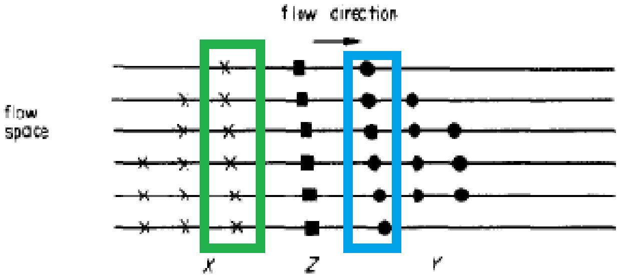
Different Perspective



Theorem

(Fokkink) Two suspensions $\mathrm{Susp}(f_i; X_i)$ *of minimal Cantor set homeomorphisms are homeomorphic, if and only if there are clopen subsets* $K_i \subset X_i$ *with respective return maps* R_i *such that either:* R_1 *is topologically conjugate to* R_2 *or* R_1 *is topologically conjugate to* R_2^{-1}.

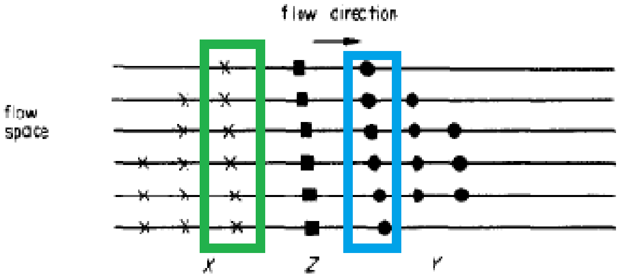
Different Perspective



Theorem

(Fokkink) Two suspensions $\mathrm{Susp}(f_i; X_i)$ *of minimal Cantor set homeomorphisms are homeomorphic, if and only if there are clopen subsets* $K_i \subset X_i$ *with respective return maps* R_i *such that either:* R_1 *is topologically conjugate to* R_2 *or* R_1 *is topologically conjugate to* R_2^{-1}.

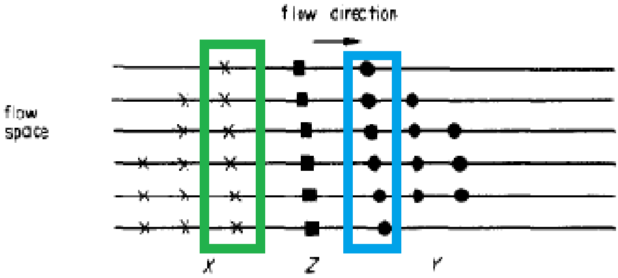
Different Perspective



Theorem

(Fokkink) Two suspensions $\mathrm{Susp}(f_i; X_i)$ *of minimal Cantor set homeomorphisms are homeomorphic, if and only if there are clopen subsets* $K_i \subset X_i$ *with respective return maps* R_i *such that either:* R_1 *is topologically conjugate to* R_2 *or* R_1 *is topologically conjugate to* R_2^{-1}*.*

Different Perspective



Theorem

(Fokkink) Two suspensions $\mathrm{Susp}(f_i; X_i)$ *of* *minimal* *Cantor set homeomorphisms are homeomorphic, if and only if there are clopen subsets* $K_i \subset X_i$ *with respective return maps* R_i *such that either:* R_1 *is topologically conjugate to* R_2 *or* R_1 *is topologically conjugate to* R_2^{-1}.

Adding Machines

Associated to a sequence of positive integers $M = (m_1, m_2, \dots)$ is the pro–finite group

$$\mathcal{A}_M = \lim_{\leftarrow} \{ \mathbb{Z}/m_1\mathbb{Z} \leftarrow \mathbb{Z}/m_2m_1\mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{Z}/m_i \cdots m_2m_1\mathbb{Z} \leftarrow \cdots \}$$

which supports the minimal homeomorphism

$$a_M : \mathcal{A}_M \longrightarrow \mathcal{A}_M$$

given by

$$a_M(x_i) = (x_i + 1 \mod m_i \cdots m_2m_1)$$

Classifying Adding Machines

Definition

Given a sequence of positive integers $M = (m_1, m_2, \dots)$, let C_M denote the function from the prime numbers to $\{0, 1, 2, ..., \infty\}$ given by

$$C_M(p) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} p_i,$$

where p_i is the power of the prime p in the prime factorization of m_i.

Theorem

(Block, Keesling 2004) The adding machines a_M and a_N are topologically conjugate if and only if $C_M = C_N$.

Classifying Adding Machines

Definition

Given a sequence of positive integers $M = (m_1, m_2, \ldots)$, let C_M denote the function from the prime numbers to $\{0, 1, 2, ..., \infty\}$ given by

$$C_M(p) \overset{\text{def}}{=} \sum_{i=1}^{\infty} p_i,$$

where p_i is the power of the prime p in the prime factorization of m_i.

Theorem

(Block, Keesling 2004) The adding machines a_M and a_N are topologically conjugate if and only if $C_M = C_N$.

Classifying Adding Machines

Definition

Given a sequence of positive integers $M = (m_1, m_2, \ldots)$, let C_M denote the function from the prime numbers to $\{0, 1, 2, ..., \infty\}$ given by

$$C_M(p) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} p_i,$$

where p_i is the power of the prime p in the prime factorization of m_i.

Theorem

(Block, Keesling 2004) The adding machines a_M and a_N are topologically conjugate if and only if $C_M = C_N$.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; A_M)$ *and* $\mathrm{Susp}(a_N; A_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; A_M)$ *and* $\mathrm{Susp}(a_N; A_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; A_M)$ *and* $\mathrm{Susp}(a_N; A_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; A_M)$ *and* $\mathrm{Susp}(a_N; A_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; \mathcal{A}_M)$ *and* $\mathrm{Susp}(a_N; \mathcal{A}_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; \mathcal{A}_M)$ *and* $\mathrm{Susp}(a_N; \mathcal{A}_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; \mathcal{A}_M)$ *and* $\mathrm{Susp}(a_N; \mathcal{A}_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Classifying Suspensions of Adding Machines

Definition

Two sequences M and N are return equivalent, denoted $M \overset{\text{Ret}}{\sim} N$, if and only if the following two conditions hold:

1. For all but finitely many primes p, $C_M(p) = C_N(p)$ and
2. for all primes p, $C_M(p) = \infty$ if and only if $C_N(p) = \infty$.

Theorem

(McCord, 1965) The suspensions $\mathrm{Susp}(a_M; \mathcal{A}_M)$ *and* $\mathrm{Susp}(a_N; \mathcal{A}_N)$ *are homeomorphic if and only if* $M \overset{\text{Ret}}{\sim} N$.

Aarts and Fokkink constructed a proof using the return maps (adding machines) to a Cantor set cross section of the related flows.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension d* is a space which admits an atlas of charts

$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension* d is a space which admits an atlas of charts

$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension* d is a space which admits an atlas of charts

$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension* d is a space which admits an atlas of charts

$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension* d is a space which admits an atlas of charts

$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

Generalizations to Higher Dimensions

Definition

A *(smooth) foliated space of dimension* d is a space which admits an atlas of charts

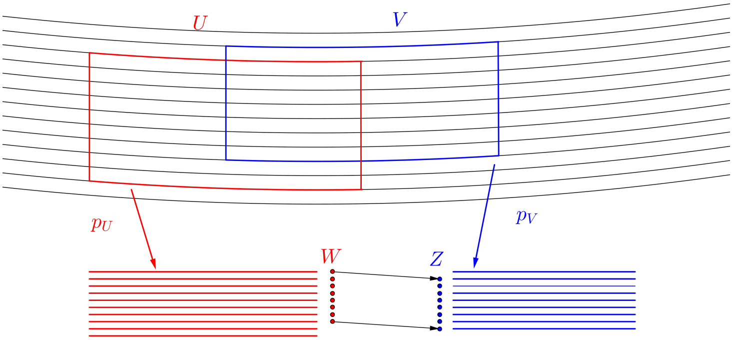
$$U \to \mathbb{R}^d \times T,$$

where each T is a subspace of the *transverse space* $\mathcal{T}$ and is such that the transition maps between charts are smooth along leaves and depend continuously on the points of transverse space and plaques in intersecting charts intersect each other in open sets.

Definition

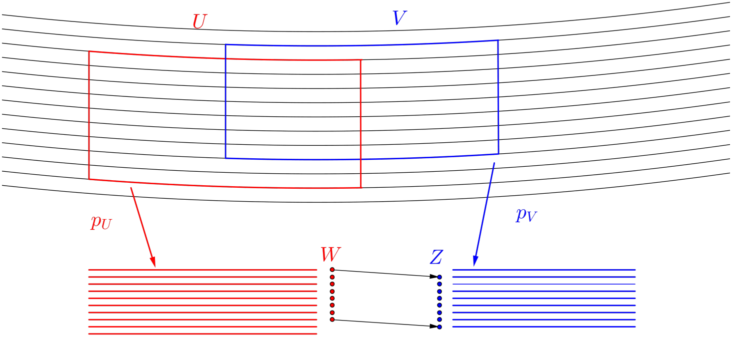
A *matchbox manifold* is a compact, connected foliated space in which the transverse space is totally disconnected.

The intersection of charts in regular foliated atlas



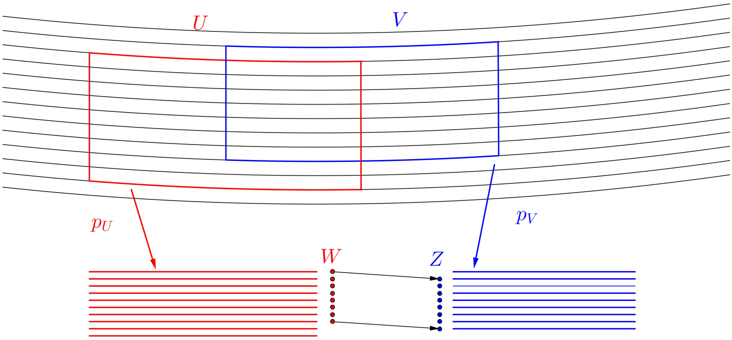
The intersection determines a homeomorphism $W \to Z$, where W is a subspace of the transversal of U and Z is a subspace of the transversal of V.

The intersection of charts in regular foliated atlas



The intersection determines a homeomorphism $W \to Z$, where W is a subspace of the transversal of U and Z is a subspace of the transversal of V.

The intersection of charts in regular foliated atlas



The intersection determines a homeomorphism $W \to Z$, where W is a subspace of the transversal of U and Z is a subspace of the transversal of V.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $id_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset dom(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $im(f) \subset dom(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_i$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_i \in \Psi$ for all i, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Holonomy Pseudogroups

Such transition maps of a regular foliated atlas generate the *holonomy pseudogroup* Ψ on the transverse space $\mathcal{T}$.

Definition

A *pseudogroup* Ψ on a topological space $\mathcal{T}$ is a collection of homeomorphisms between open subsets of $\mathcal{T}$ satisfying:

- $\mathrm{id}_{\mathcal{T}} \in \Psi$
- *(Restriction)* For any given $f \in \Psi$ and open $U \subset \mathrm{dom}(f)$, $f|U \in \Psi$
- *(Inverses)* If $f \in \Psi$, then $f^{-1} \in \Psi$
- *(Composition)* If for $f, g \in \Psi$, $\mathrm{im}(f) \subset \mathrm{dom}(g)$, then $g \circ f \in \Psi$
- *(Amalgamation)* If for open $U = \cup U_\alpha$ and homeomorphism $f : U \to V$ between open sets, we have $f|U_\alpha \in \Psi$ for all α, then $f \in \Psi$.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_i of $\mathcal{M}_i$ there are clopen subsets $C_i \subset T_i$ such that the pseudogroup of return holonomy maps to C_i are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_j of $\mathcal{M}_j$ there are clopen subsets $C_j \subset T_j$ such that the pseudogroup of return holonomy maps to C_j are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_j of $\mathcal{M}_j$ there are clopen subsets $C_j \subset T_j$ such that the pseudogroup of return holonomy maps to C_j are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_j of $\mathcal{M}_j$ there are clopen subsets $C_j \subset T_j$ such that the pseudogroup of return holonomy maps to C_j are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_j of $\mathcal{M}_j$ there are clopen subsets $C_j \subset T_j$ such that the pseudogroup of return holonomy maps to C_j are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

Return Equivalence

For a given clopen subset C of a transversal T in a regular foliated atlas of our matchbox manifold, we then consider the induced holonomy pseudogroup generated by the holonomy maps from open subsets of C to open subsets of C.

Definition

Two matchbox manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if for any given pair of transversals T_j of $\mathcal{M}_j$ there are clopen subsets $C_j \subset T_j$ such that the pseudogroup of return holonomy maps to C_j are conjugate.

Theorem

(C., Hurder, Lukina)(2013) The relation of being return equivalent defines an equivalence relation on minimal matchbox manifolds. Moreover, any two homeomorphic minimal matchbox manifolds are return equivalent.

General Adding Machines and their Suspensions

Let M be a closed manifold with fundamental group $\pi_1(M)$. Given a profinite "almost" group of the form

$$\mathcal{C} = \varprojlim \{ \pi_1(M)/G_1 \leftarrow \pi_1(M)/G_2 \leftarrow \cdots \leftarrow \pi_1(M)/G_i \leftarrow \cdots \}$$

where $G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ are (not necessarily normal) subgroups of $\pi_1(M)$ of finite index, there is a natural minimal action of $\pi_1(M)$ on $\mathcal{C}$ given by translation in each factor.

By taking the orbit space of the action of $\pi_1(M)$ on $\widetilde{M} \times \mathcal{C}$, one forms the suspension

$$\widetilde{M} \times_{\pi_1(M)} \mathcal{C}$$

which is a minimal matchbox manifold.

Theorem

(C., Hurder) All equicontinuous matchbox manifolds are of this form.

General Adding Machines and their Suspensions

Let M be a closed manifold with fundamental group $\pi_1(M)$. Given a profinite "almost" group of the form

$$\mathcal{C} = \lim_{\leftarrow} \{ \pi_1(M)/G_1 \leftarrow \pi_1(M)/G_2 \leftarrow \cdots \leftarrow \pi_1(M)/G_i \leftarrow \cdots \}$$

where $G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ are (not necessarily normal) subgroups of $\pi_1(M)$ of finite index, there is a natural minimal action of $\pi_1(M)$ on $\mathcal{C}$ given by translation in each factor.

By taking the orbit space of the action of $\pi_1(M)$ on $\widetilde{M} \times \mathcal{C}$, one forms the suspension

$$\widetilde{M} \times_{\pi_1(M)} \mathcal{C}$$

which is a minimal matchbox manifold.

Theorem

(C., Hurder) All equicontinuous matchbox manifolds are of this form.

General Adding Machines and their Suspensions

Let M be a closed manifold with fundamental group $\pi_1(M)$. Given a profinite "almost" group of the form

$$\mathcal{C} = \varprojlim \{\pi_1(M)/G_1 \leftarrow \pi_1(M)/G_2 \leftarrow \cdots \leftarrow \pi_1(M)/G_i \leftarrow \cdots\}$$

where $G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ are (not necessarily normal) subgroups of $\pi_1(M)$ of finite index, there is a natural minimal action of $\pi_1(M)$ on $\mathcal{C}$ given by translation in each factor.

By taking the orbit space of the action of $\pi_1(M)$ on $\widetilde{M} \times \mathcal{C}$, one forms the suspension

$$\widetilde{M} \times_{\pi_1(M)} \mathcal{C}$$

which is a minimal matchbox manifold.

Theorem

(C., Hurder) All equicontinuous matchbox manifolds are of this form.

General Adding Machines and their Suspensions

Let M be a closed manifold with fundamental group $\pi_1(M)$. Given a profinite "almost" group of the form

$$\mathcal{C} = \lim_{\leftarrow} \{ \pi_1(M)/G_1 \leftarrow \pi_1(M)/G_2 \leftarrow \cdots \leftarrow \pi_1(M)/G_i \leftarrow \cdots \}$$

where $G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ are (not necessarily normal) subgroups of $\pi_1(M)$ of finite index, there is a natural minimal action of $\pi_1(M)$ on $\mathcal{C}$ given by translation in each factor.

By taking the orbit space of the action of $\pi_1(M)$ on $\widetilde{M} \times \mathcal{C}$, one forms the suspension

$$\widetilde{M} \times_{\pi_1(M)} \mathcal{C}$$

which is a minimal matchbox manifold.

Theorem

(C., Hurder) All equicontinuous matchbox manifolds are of this form.

General Adding Machines and their Suspensions

Let M be a closed manifold with fundamental group $\pi_1(M)$. Given a profinite "almost" group of the form

$$\mathcal{C} = \varprojlim \{ \pi_1(M)/G_1 \leftarrow \pi_1(M)/G_2 \leftarrow \cdots \leftarrow \pi_1(M)/G_i \leftarrow \cdots \}$$

where $G_1 \supset G_2 \supset \cdots \supset G_i \supset \cdots$ are (not necessarily normal) subgroups of $\pi_1(M)$ of finite index, there is a natural minimal action of $\pi_1(M)$ on $\mathcal{C}$ given by translation in each factor.

By taking the orbit space of the action of $\pi_1(M)$ on $\widetilde{M} \times \mathcal{C}$, one forms the suspension

$$\widetilde{M} \times_{\pi_1(M)} \mathcal{C}$$

which is a minimal matchbox manifold.

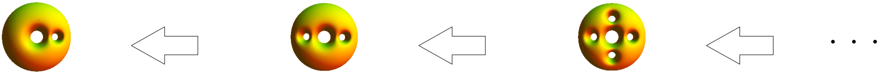
Theorem

(C., Hurder) All equicontinuous matchbox manifolds are of this form.

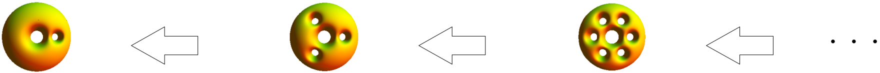
Adic Surfaces

We call the suspension of a standard adding machine over a surface an *adic surface*. Examples over a surface of genus 2:

S_M for $M = (2, 2, ...)$



S_N for $N = (3, 2, ...)$



The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\mathrm{pro} - \pi_1$ and a consideration of the Euler characteristics.

The Classification of Adic Surfaces

Proposition

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are return equivalent if and only if $M \overset{\text{Ret}}{\sim} N$.

This follows from the corresponding result in dimension one.

Theorem

(C., Hurder, Lukina) (2013)
The adic surfaces S_M and S_N are homeomorphic if and only if $C_M = C_N$.

The sufficiency of $C_M = C_N$, follows from Block and Keesling's criterion for the conjugacy of adding machines.
For the other direction, one uses the diagram in $\text{pro} - \pi_1$ and a consideration of the Euler characteristics.

Return equivalence yielding topological classification

Theorem

(C., Hurder, Lukina) (2013)
Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are equicontinuous $\mathbb{T}^n$-like matchbox manifolds.
Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are homeomorphic.

We have a version of this theorem for a more general class of manifolds to which the Borel conjecture applies, provided that the leaves are *simply connected*. The key point in this situation is that the information about the π_1 action encodes at the same time the homotopy information of the manifold factors.

Return equivalence yielding topological classification

Theorem

(C., Hurder, Lukina) (2013)
Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are equicontinuous $\mathbb{T}^n$-like matchbox manifolds.
Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are homeomorphic.

We have a version of this theorem for a more general class of manifolds to which the Borel conjecture applies, provided that the leaves are *simply connected*. The key point in this situation is that the information about the π_1 action encodes at the same time the homotopy information of the manifold factors.

Return equivalence yielding topological classification

Theorem

(C., Hurder, Lukina) (2013)
Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are equicontinuous $\mathbb{T}^n$-like matchbox manifolds.
Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are homeomorphic.

We have a version of this theorem for a more general class of manifolds to which the Borel conjecture applies, provided that the leaves are *simply connected*. The key point in this situation is that the information about the π_1 action encodes at the same time the homotopy information of the manifold factors.

Return equivalence yielding topological classification

Theorem

(C., Hurder, Lukina) (2013)
Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are equicontinuous $\mathbb{T}^n$-like matchbox manifolds.
Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are return equivalent if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are homeomorphic.

We have a version of this theorem for a more general class of manifolds to which the Borel conjecture applies, provided that the leaves are *simply connected*. The key point in this situation is that the information about the π_1 action encodes at the same time the homotopy information of the manifold factors.

Tiling Spaces and Return Equivalence

We have yet to fully investigate the role of return equivalence in tiling spaces.

Theorem

(Fokkink; Barge, Williams) The suspensions of the Sturmian minimal sets associated to α and β are homeomorphic if and only if their continued fraction expansions share a common tail.

Conjecture

Two non–periodic translational tiling spaces of finite local complexity are homeomorphic if they are return equivalent.

Tiling Spaces and Return Equivalence

We have yet to fully investigate the role of return equivalence in tiling spaces.

Theorem

(Fokkink; Barge, Williams) The suspensions of the Sturmian minimal sets associated to α and β are homeomorphic if and only if their continued fraction expansions share a common tail.

Conjecture

Two non–periodic translational tiling spaces of finite local complexity are homeomorphic if they are return equivalent.

Tiling Spaces and Return Equivalence

We have yet to fully investigate the role of return equivalence in tiling spaces.

Theorem

(Fokkink; Barge, Williams) The suspensions of the Sturmian minimal sets associated to α and β are homeomorphic if and only if their continued fraction expansions share a common tail.

Conjecture

Two non–periodic translational tiling spaces of finite local complexity are homeomorphic if they are return equivalent.

Tiling Spaces and Return Equivalence

We have yet to fully investigate the role of return equivalence in tiling spaces.

Theorem

(Fokkink; Barge, Williams) The suspensions of the Sturmian minimal sets associated to α and β are homeomorphic if and only if their continued fraction expansions share a common tail.

Conjecture

Two non–periodic translational tiling spaces of finite local complexity are homeomorphic if they are return equivalent.

Tiling Spaces and Return Equivalence

We have yet to fully investigate the role of return equivalence in tiling spaces.

Theorem

(Fokkink; Barge, Williams) The suspensions of the Sturmian minimal sets associated to α and β are homeomorphic if and only if their continued fraction expansions share a common tail.

Conjecture

Two non–periodic translational tiling spaces of finite local complexity are homeomorphic if they are return equivalent.