Current fluctuations in the two-dimensional zero-range process

Rodrigo Villavicencio Sánchez

Queen Mary University of London

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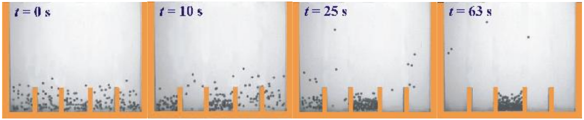

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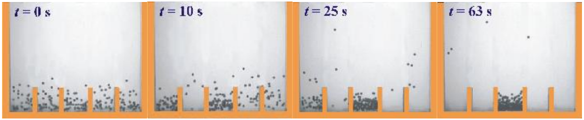

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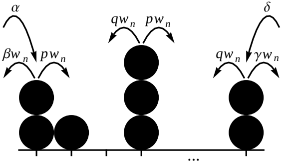
- Understanding systems out of equilibrium.

Zero-range process

- Particles accumulate on each site of the lattice up to any non-negative number n.



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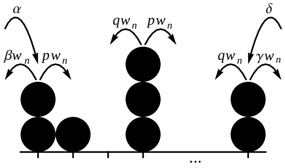

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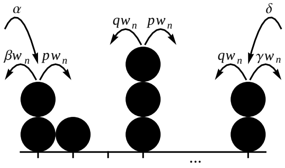

- The topmost particle of a site jumps after an exponentially distributed waiting time.
- The time evolution of the ZRP is given by the master equation:

$$\frac{d|P\rangle}{dt} = -H|P\rangle$$

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In long-time limit expect the time-averaged current, $j_{a,b}$, between sites a and b follow a large deviation principle:

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How to measure the fluctuations

$e(\lambda)$ is the lowest eigenvalue of a modified Hamiltonian.

The Gallavotti-Cohen fluctuation relation (GCFR) provides a well-known symmetry for the entropy production rate in non-equilibrium systems. [Evans, Gallavotti & Cohen, $\sim$1993]

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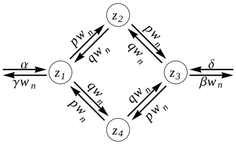
In terms of the GFs

$$e(E+\lambda) = e(E-\lambda)$$

Quick application:

- We study the appearance of current loops within diamond lattice.



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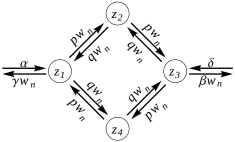
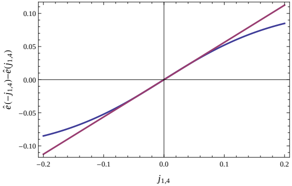


- The existence of current loops implies breakdown for the GCFR for partial currents, in agreement with other papers.



- R. Villavicencio-Sanchez, R. J. Harris, H. Touchette, arXiv: 1202.3989.

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- The large deviation principle for particle currents in d-dimensions:

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Isometric fluctuation relation [Hurtado et al., 2011]:

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\frac{p(\mathbf{j}, t)}{p(\mathbf{j}', t)}\right) = \mathbf{E} \cdot (\mathbf{j} - \mathbf{j}') \text{ with } |\mathbf{j}| = |\mathbf{j}'|$$

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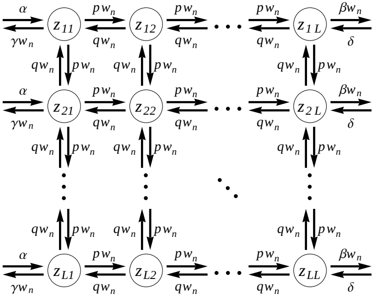
- For the ZRP $D = z'(\rho)$ and $\sigma = z(\rho)$ [Bertini et al., 2007]. e.g. For non-interacting particles $D = 1$ and $\sigma = \rho$.

Back to the microscopic approach

- The modified Hamiltonian for the $L \times L$ lattice:



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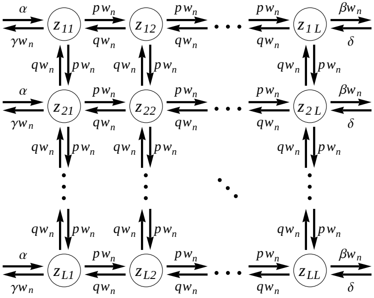

- The SCGF:

$$e(\boldsymbol{\lambda}) = \sum_{i=1}^{L} \left\{ \alpha + \delta - \left(\gamma e^{\lambda_x} z_{i1} + \beta e^{-\lambda_x} z_{iL} \right) \right\}$$

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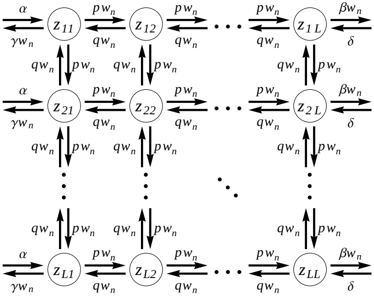

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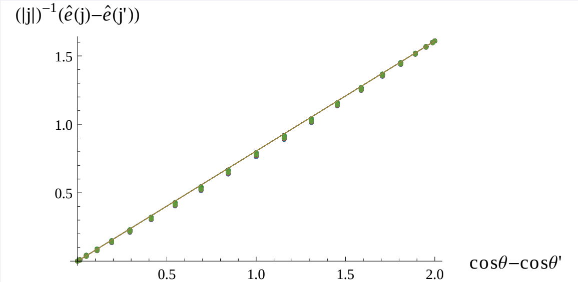
- Legendre transform

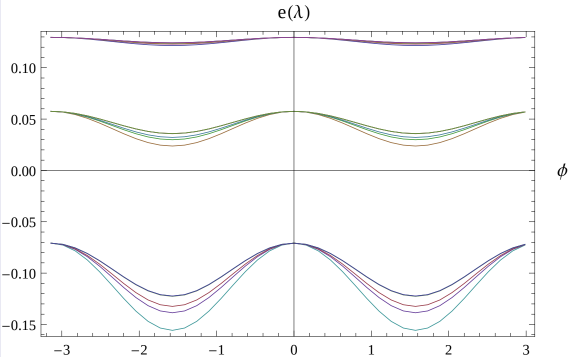
From the large deviation principle:

$$\hat{e}(\mathbf{j}) - \hat{e}(\mathbf{j}') = |\mathbf{E}||\mathbf{j}|(\cos\theta - \cos\theta')$$



The IFR implies that for equidistant fluctuations from the maximum, the SCGF should be constant:

$$e(\boldsymbol{\lambda}) = e(\mathcal{R}_\phi(\boldsymbol{\lambda} + \mathbf{E}) - \mathbf{E})$$



Overview

- The ZRP offers the possibility to analytically test fluctuation symmetries in higher dimensions.
- The GCFR needs to be measured for total currents. (Current loops cause breakdown for partial currents).
- We see IFR holds for hydrodynamic limit with convergence to it for large lattice sizes.
- Outlook:
  - Check hydrodynamic limit for other w_n.
  - Study the asymmetric case (i.e. not purely diffusive).

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