

Anomalous Fluctuation Relations

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Outline

- **'Normal' fluctuation relations:**
motivation and warm-up

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check transient fluctuation relations for correlated Gaussian stochastic dynamics
- **Relations to experiments:**
glassy dynamics and cell migration

Motivation: Fluctuation relations

Consider a (classical) particle system evolving from some initial state into a nonequilibrium steady state.

Measure the probability distribution $\rho(\xi_t)$ of entropy production ξ_t during time t :

$$\ln \frac{\rho(\xi_t)}{\rho(-\xi_t)} = \xi_t$$

Transient Fluctuation Relation (TFR)

Evans, Cohen, Morriss (1993); Gallavotti, Cohen (1995)

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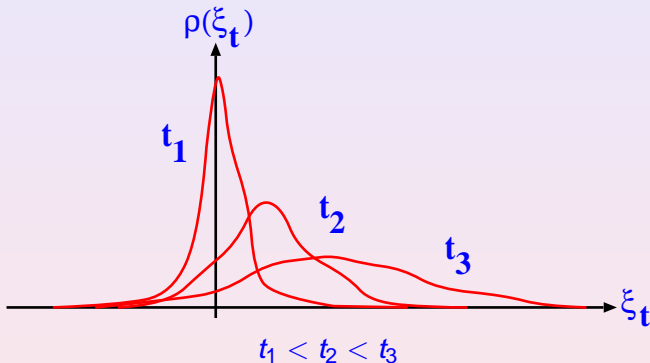
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why important? of *very general validity* and

- 1 generalizes the **Second Law** to small noneq. systems
- 2 connection with **fluctuation dissipation relations**
- 3 can be checked in **experiments** (Wang et al., 2002)

Fluctuation relation and the Second Law

meaning of TFR in terms of the Second Law:



$$\rho(\xi_t) = \rho(-\xi_t) \exp(\xi_t) \geq \rho(-\xi_t) \quad (\xi_t \geq 0) \Rightarrow \langle \xi_t \rangle \geq 0$$

sample specifically the tails of the pdf (large deviation result)

Fluctuation relation for Langevin dynamics

warmup: check TFR for the overdamped **Langevin equation**

$$\dot{x} = F + \zeta(t) \quad (\text{set all irrelevant constants to 1})$$

with **constant field** F and Gaussian white noise $\zeta(t)$.

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entropy production ξ_t is equal to (mechanical) **work** $W_t = Fx(t)$

with $\rho(W_t) = F^{-1} \varrho(x, t)$; remains to solve corresponding

Fokker-Planck equation for initial condition $x(0) = 0$:

the position pdf is Gaussian,

$$\varrho(x, t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(x - \langle x \rangle)^2}{2\sigma_x^2}\right)$$

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straightforward:

$$(\text{work}) \text{ TFR holds if } \langle x \rangle = \sigma_x^2/2$$

and \exists **fluctuation-dissipation relation 1 (FDR1) \Rightarrow TFR**

see, e.g., **van Zon, Cohen, PRE (2003)**

Gaussian stochastic dynamics

goal: check TFR for **Gaussian stochastic processes** defined by the (overdamped) **generalized Langevin equation**

$$\int_0^t dt' \dot{x}(t') K(t-t') = F + \zeta(t)$$

e.g., **Kubo (1965)**

with **Gaussian noise** $\zeta(t)$ and **memory kernel** $K(t)$

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such dynamics can generate **anomalous diffusion**:

$$\sigma_x^2 \sim t^\alpha \text{ with } \alpha \neq 1 \text{ (} t \rightarrow \infty \text{)}$$

examples of applications: polymer dynamics (**Panja, 2010**);
biological cell migration (**Dieterich et al., 2008**)

TFR for correlated internal Gaussian noise

consider two generic cases:

1. **internal Gaussian noise** defined by the **FDR2**,

$$\langle \zeta(t)\zeta(t') \rangle \sim K(t-t'),$$

with **non-Markovian (correlated) noise**; e.g., $K(t) \sim t^{-\beta}$

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and since $\rho(W_t) \sim \varrho(x, t)$ is Gaussian

$$\text{'FDR1'} \Rightarrow \text{TFR}$$

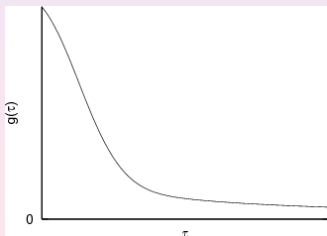
for correlated internal Gaussian noise \exists TFR

Correlated external Gaussian noise

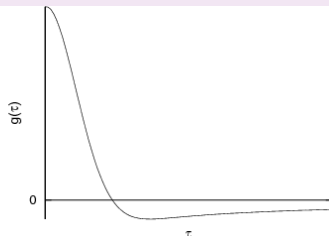
2. external Gaussian noise for which there is **no FDR2**, modeled by the (overdamped) generalized Langevin equation

$$\dot{x} = F + \zeta(t)$$

consider two types of **Gaussian noise correlated** by $g(\tau) = \langle \zeta(t)\zeta(t') \rangle_{\tau=t-t'} \sim (\Delta/\tau)^\beta$ for $\tau > \Delta$, $\beta > 0$:



persistent



anti-persistent

it is $\langle x \rangle = Ft$ and $\sigma_x^2 = 2 \int_0^t d\tau (t - \tau)g(\tau)$

TFRs for correlated external Gaussian noise I

persistent noise:

results for σ_x^2 and the **fluctuation ratio** $R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)}$

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- $0 < \beta < 1$:

superdiffusion $\sigma_x^2 \sim t^{2-\beta}$ with anomalous TFR $R \sim \frac{W_t}{t^{1-\beta}}$

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- $\beta = 1$:

weak superdiffusion $\sigma_x^2 \sim t \ln \left(\frac{t}{\Delta} \right)$ with **weakly anomalous TFR**

$$R \sim W_t / \ln \left(\frac{t}{\Delta} \right)$$

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- $1 < \beta < \infty$:

normal diffusion $\sigma_x^2 \sim 2Dt$ with $D = \int_0^\infty d\tau g(\tau)$ and **anomalous (generalized) TFR** $R \sim \frac{W_t}{D}$

TFRs for correlated external Gaussian noise II

antipersistent noise:

$\int_0^\infty d\tau g(\tau) > 0$ yields **normal diffusion** with a **generalized TFR**

for $t \gg \Delta$; for 'pure' antipersistent case with $\int_0^\infty d\tau g(\tau) = 0$:

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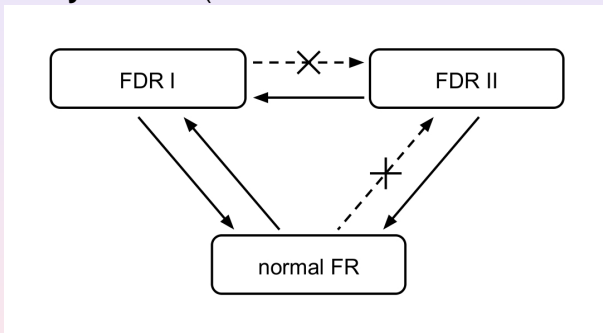
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 $R \sim W_t t / \ln(t/\Delta)$
- $2 < \beta < \infty$:
localization $\sigma_x^2 = \text{const.}$ with **anomalous TFR** $R \sim W_t t$

FDR and TFR

relation between **TFR** and **FDR I,II** for **correlated Gaussian stochastic dynamics**: ('normal FR'= conventional TFR)



in particular:

$$\boxed{\text{FDR2} \Rightarrow \text{FDR1} \Rightarrow \text{TFR}}$$

$$\boxed{\nexists \text{TFR} \Rightarrow \nexists \text{FDR2}}$$

Relations to experiments: A glassy lattice gas

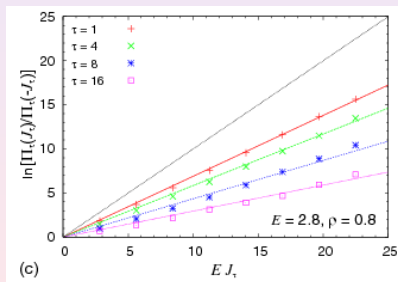
$$R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)} = \mathbf{f}_\beta(\mathbf{t}) W_t$$

means by plotting R for different t the **slope should change**.

example 1:

computer simulations
for glassy lattice gas
with external field E

Sellitto, PRE (2009)

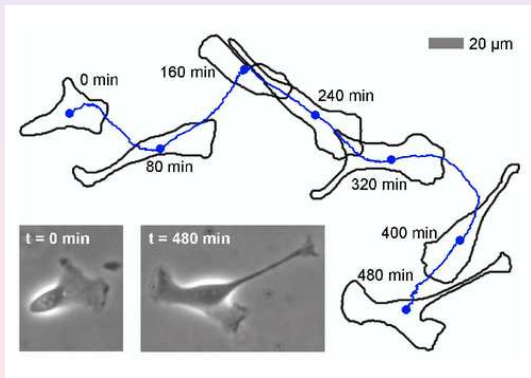


similar simulation results for three other models exhibiting
glassy dynamics: Crisanti et al., PRL (2013)

Biological cell migration

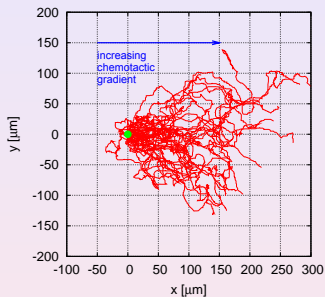
example 2:

single biological cell crawling on a substrate; trajectory recorded with a video camera (Dieterich et al., 2008)



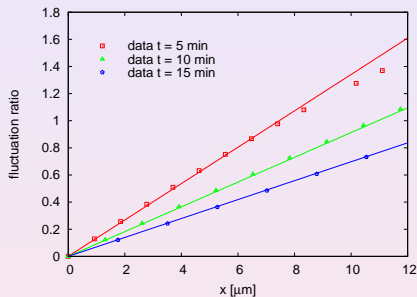
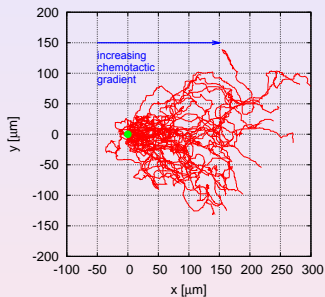
Cell migration under chemical gradients

experiments on **murine neutrophils** under **chemotaxis**:



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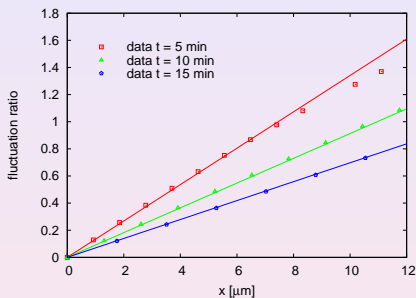
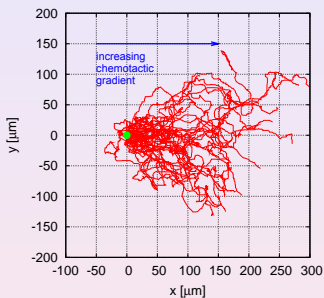
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- **linear drift** in the direction of the gradient, $\langle x(t) \rangle \sim t$
- $\sigma_x^2 \sim t^\alpha$ with $\alpha > 1$ (long t): \nexists FDR1
- some relation to the **generalized Langevin equation** with external noise and $0 < \beta < 1$ discussed before

Summary

- TFR tested for two generic cases of **correlated Gaussian stochastic dynamics**:
 - 1 **internal noise**:
FDR2 implies the validity of the 'normal' work TFR
 - 2 **external noise**:
FDR2 is broken; sub-classes of **persistent** and **anti-persistent noise** yield both **anomalous TFRs**

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 - 2 **external noise**:
FDR2 is broken; sub-classes of **persistent** and **anti-persistent noise** yield both **anomalous TFRs**
- anomalous TFRs appear to be important for **glassy dynamics**: cf. computer simulations on various glassy models and experiments on ('gelly') cell migration

References

- **A.V. Chechkin, F.Lenz, RK, J. Stat. Mech. L11001 (2012)**
- **A.V. Chechkin, RK, J. Stat. Mech. L03002 (2009)**
- **RK, A.V. Chechkin, P.Dieterich, *Anomalous fluctuation relations* in:**

