Numerical Real-Space Renormalisation Transformations on Finite Ising and Potts Lattices in 2 Dimensions

Gary Willis

Imperial College London

gary.willis07@imperial.ac.uk

26 March 2014

What is the Real-Space Renormalisation Group (RSRG) ?

$$
\begin{array}{cccc}
+ & + & - & - \\
+ & + & - & + \\
+ & - & + & - \\
+ & + & + & -
\end{array}
\longrightarrow
\begin{array}{cc}
+ & - \\
+ & +
\end{array}
$$

What can the RSRG tell us?

- It provides a framework for calculating critical exponents
- $y_t = \frac{1}{\nu}$
- $\xi \sim t^{-\nu}$
- $y_h = \frac{d}{1+\frac{1}{\delta}}$
- $m(t=0,h) \sim sgn(h)|h|^{\frac{1}{\delta}}$

A tiny bit of maths...

$$Z = Z'$$

$$\sum_{\{s_i\}} e^{-H(s_i)} = \sum_{\{S_I\}} e^{-H'(S_I)}$$

$$\sum_{\{S_I\}} \sum_{\{s_i\} \to \{S_i\}} e^{-H(s_i)} = \sum_{\{S_I\}} e^{-H'(S_I)}$$

$$\sum_{\{s_i\} \to \{S_i\}} e^{-H(s_i)} = e^{-H'(S_I)}$$

The LHS is a long sum which we then evaluate numerically

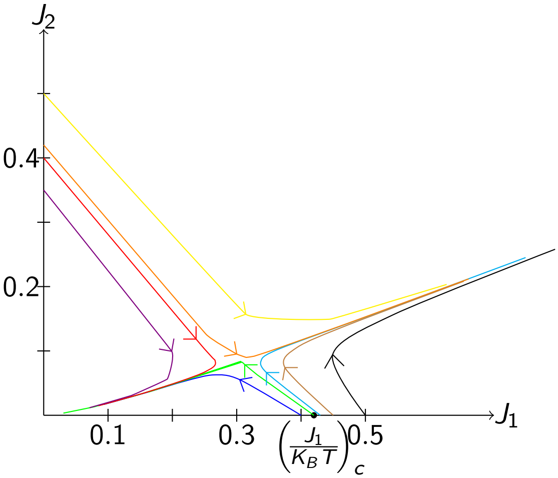
The Ising Model and Hasenbusch's Method

- The Ising Model States $H(s_i) = -J \sum_{\langle ij \rangle} s_i s_j$
- Demanding $Z = Z'$ will generate a new Hamiltonian with new terms in it. General Ising-like Hamiltonian might look like:

$$H'(S_I) = -J'_0 - J'_1 \sum_{\langle ij \rangle} S_I S_J - J'_2 \sum_{\langle\langle ij \rangle\rangle} S_I S_J - J'_3 \sum_{\square} S_I S_J S_K S_L + \dots \quad (1)$$

- Hasenbusch performs the RSRG from a $4 \times 4 \longrightarrow 2 \times 2$ lattice
- Equation (1) is the most general Hamiltonian which respects rotational and parity symmetry on a 2×2 lattice.

Results



y_t is obtained to be $\simeq 0.936$ and thus $\nu \simeq 1.068$ or an error of $\sim 7\%$ against Onsager's analytical value.

Extending to Allow for an External Field

- Previously, Equation (1) did not contain terms of the form $J_5 \sum_{\langle IJK \rangle} S_I S_J S_K$ and $J_4 \sum_I S_I$
- When such terms are allowed, their coupling constants only renormalise to non zero values if they are initially non-zero
- The fixed point remains unchanged, i.e. the magnetic components are zero
- One obtains a further relevant direction, and $y_h = 1.828$ (analytically $\frac{15}{8} = 1.875$ and thus only $\sim 2.5\%$ error)

The Categories Method

C_0	$++$ $++$	$--$ $--$						
C_1	$++$ $+-$	$++$ $-+$	$+-$ $++$	$-+$ $++$	$--$ $-+$	$--$ $+-$	$-+$ $--$	$+-$ $--$
C_2	$++$ $--$	$+-$ $+-$	$-+$ $-+$	$--$ $++$				
C_3	$+-$ $-+$	$-+$ $+-$						

The 4 different values which can be obtained for the sixteen configurations of a 2×2 lattice when Evaluating Equation (1).

The Categories Method(2)

C_0	$++$ $++$			
C_1	$--$ $--$			
C_2	$++$ $+-$	$++$ $-+$	$+-$ $++$	$-+$ $++$
C_3	$--$ $-+$	$--$ $+-$	$-+$ $--$	$+-$ $--$
C_4	$++$ $--$	$+-$ $+-$	$-+$ $-+$	$--$ $++$
C_5	$+-$ $-+$	$-+$ $+-$		

- If one allows for magnetic interactions, there are now six possible outcomes.

The Potts Model

- Definition:

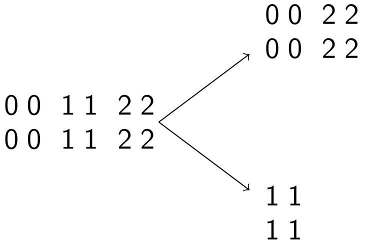
$$H = -J \sum_{\langle ij \rangle} \delta(s_i, s_j) \quad (s_i = \{1, 2, \ldots q\}) \tag{2}$$

- Now we have no "physical" Hamiltonian to guide us
- Simplest rule: "*Each configuration in a given category can be transformed into any other in the same category by a combination of operations of the symmetry which the model respects.*"

$$\begin{matrix} 1 & 2 \\ 1 & 0 \end{matrix} \longrightarrow \begin{matrix} 2 & 0 \\ 2 & 1 \end{matrix} \longrightarrow \begin{matrix} 0 & 1 \\ 2 & 2 \end{matrix}$$

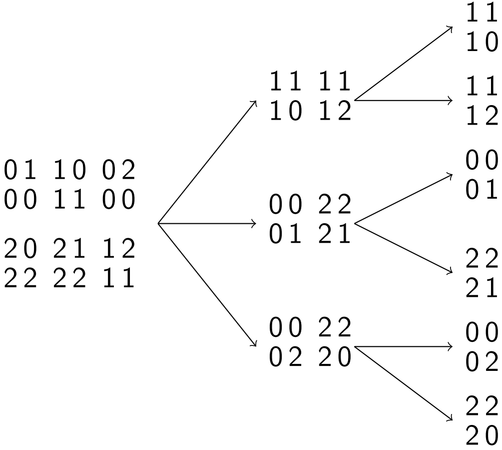
C_0	C_1			C_2			C_3	C_4			C_5		
00 00	00 01	00 02	11 12	00 11	11 22	22 00	01 10	00 12	11 02	22 01	02 10	12 01	21 02
11 11	00 10	00 20	11 21	01 01	12 12	20 20	10 01	00 21	11 20	22 10	01 20	10 21	20 12
22 22	10 00	20 00	21 11	11 00	22 11	00 22	02 20	01 02	10 12	21 20	10 02	21 10	12 20
	01 00	02 00	12 11	10 10	21 21	02 02	20 02	02 01	12 10	20 21	20 01	01 12	02 21
	11 10	22 20	22 21				12 21	12 00	20 11	01 22			
	11 01	22 02	22 12				21 12	21 00	02 11	10 22			
	01 11	02 22	12 22					10 20	01 21	12 02			
	10 11	20 22	21 22					20 10	21 01	02 12			

Introducing a Magnetic Field



Here we break Z_3 symmetry but a Z_2 symmetry remains.

Introducing a Second Magnetic Field



The first set of arrows breaks Z_3 symmetry and the second breaks Z_2.

Potts Model Results

Good Results

- $y_t = 1.118$ (analytical $=1.2$)
- $y_h = 1.810$ (analytical $=1.867$)

"Bad" Results

- A further, unexplained critical exponent is found after breaking z_3 symmetry, $y_? = 0.614$. When breaking z_2 symmetry, no new exponents are found.

Further Work

- There are different ways of choosing categories. As long as they respect the symmetries of the model, the critical exponents must be the same. Other details however such as the critical temperature must not necessarily however.
- Despite some (in my opinion very interesting) technicalities, the method can be readily extended to the triangular lattice. As expected due to universality, very similar critical exponents are obtained including $y_?$.

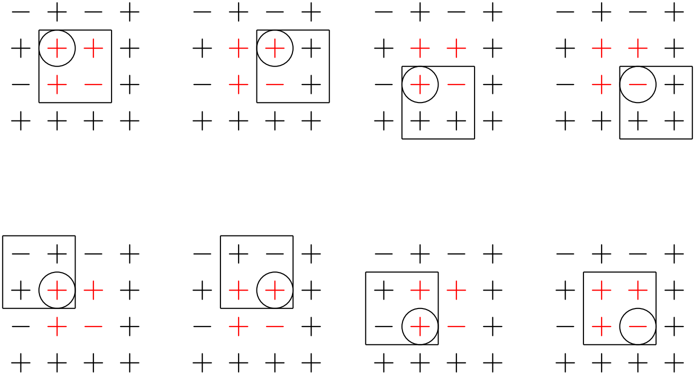
Thanks for your attention

M.Hasenbusch
Finite lattice rg-transformation for the 2d ising model

K.Christensen and N.Moloney (2005)
Complexity and Criticality, Imperial College Press

F.Y.Wu (1982)
The Potts Model, Institut für Festkörperforschung der Kernforschungsanlage Jülich

Extra Slides (1)



Extra Slides(2)