Condensation in Totally Asymmetric Inclusion Process

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March 27, 2013

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Outline

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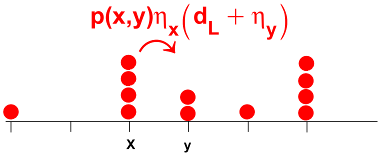
2. Condensation in TASIP Model
3. Dynamics of Condensation
   - Stationary Regime
   - Saturation Regime
   - Coarsening Regime
   - Nucleation Regime

Totally Asymmetric Inclusion Process (TASIP)



Lattice : $\Lambda_L = \{1, 2, 3, ..., L\}$ with periodic boundary condition
State space : $\mathbf{X} = \{0, 1, 2, ...\}^{\Lambda_L}$

Configuration : $\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda_L}$. Conserved particles: $\sum_{x\in\Lambda_L} \eta_x = \rho_L L = N$

$$\textbf{Generator} : \mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda} p(x,y)\eta_x(d_L + \eta_y)(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}))$$

where $p(x,y) = \begin{cases} 1, & \text{if } y = x+1 \\ 0, & \text{otherwise} \end{cases}$ (nearest neighbor jump).

Stationary product measure [Grosskinsky, Redig, Vafayi, 2011]

Condensation in TASIP model

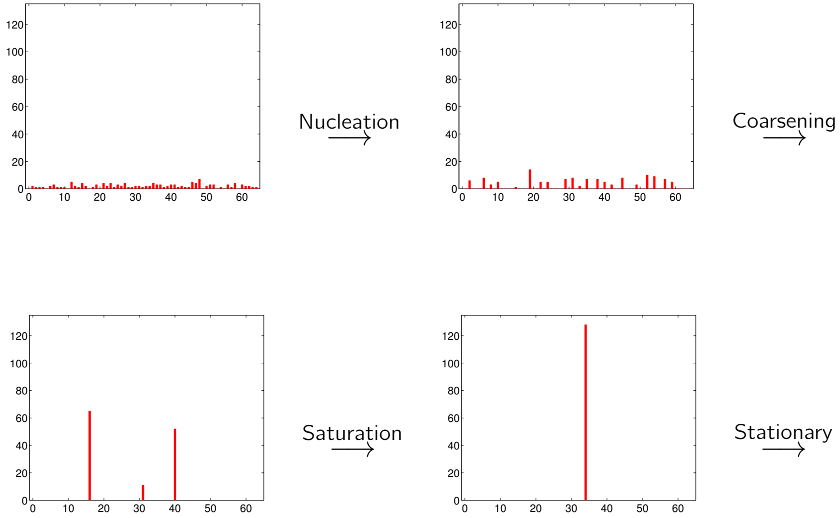
Condensation : all particles accumulate on a single site.

In the limit $d \to 0$ (weak diffusion), a condensation phenomenon occurs in inclusion process :

- L and N are both fixed. [Grosskinsky, Redig, Vafayi. 2011]
- in the limit $L, N \to \infty$, $\frac{N}{L} \to \rho > 0$. [Chleboun. 2011]
- in the limit L fixed, $N \to \infty$. [Grosskinsky, Redig, Vafayi. 2012]

MOVIE $[L = 64,\ \rho_L = 2,\ d_L = \frac{1}{L^2}]$

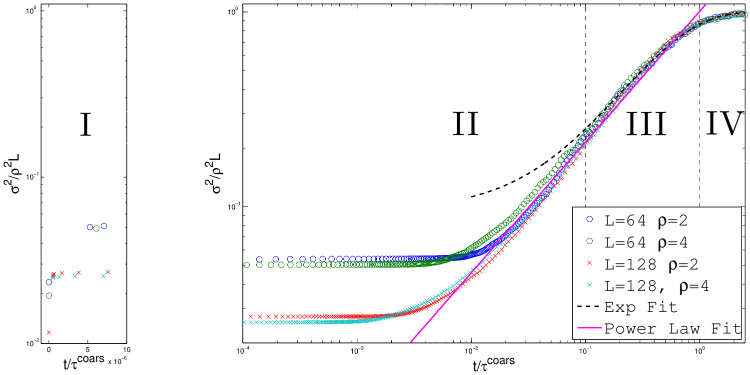
Dynamics of Condensation



Observable: $\sigma^2(t) = \mathbb{E}[\boldsymbol{\eta}^2] = \frac{1}{L}\sum_{x\in\Lambda_L} \eta_x^2$

- $\mathbb{E}[\boldsymbol{\eta}] = \rho$
- $\sigma^2(0) = \rho^2 + \rho - \frac{1}{L}\rho$ (converges to $\rho(1+\rho)$ as $L \to \infty$).
- $\sigma^2(t) \overset{t\to\infty}{\longrightarrow} \rho^2 L$

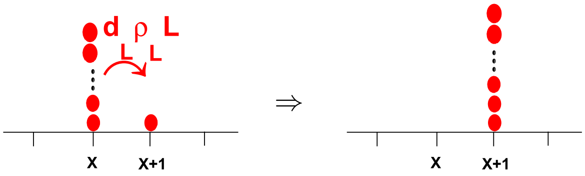
Dynamics of Condensation



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations

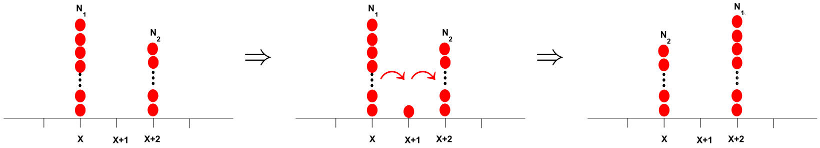
I : Nucleation Regime. II : Coarsening Regime.
III : Saturation Regime. IV : Stationary Regime.

Stationary Regime



- A particle jumps by **diffusion** with rate $d_L \rho_L L$.
- Other particles on site x follow immediately by **inclusion**.
- The single condensate moves ballistically with speed $d_L \rho_L L$.

Two condensates interaction



- N_1, N_2 are both in order L.
- Large condensate will **penetrate** small one without macroscopic change of number of particles.

Saturation Regime

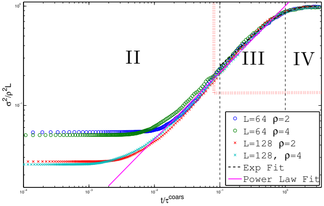
Exponential Approximation of $\sigma^2(t)$

$$\frac{d}{dt}\mathbb{E}\left[f(\boldsymbol{\eta}_t)\right] = \mathbb{E}[(\mathcal{L}_L f)(\boldsymbol{\eta}_t)] \text{ , take } f(\boldsymbol{\eta}) = \eta_z^2 \text{ , } z \in \Lambda_L .$$

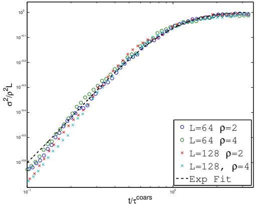
Assumption : $\eta_{z-1} = \eta_{z+1} = \frac{\rho_L L - \eta_z}{L-1}$

$\Rightarrow \sigma^2(t) \simeq \rho_L^2 L \left(1 - e^{-\frac{2}{L}t}\right)$

Saturation Regime



$\Rightarrow$



Exponential fit of saturation regime

Coarsening Regime

$n(t)$: **number** of piles. $m(t)$: **size** of a typical pile. $(m(t)n(t) = \rho_L L)$

$v \sim d_L m(t)$: **speed** of a pile. $s \sim \frac{L}{n(t)}$: average **distance** of two piles.

Differential equation

$$\frac{d}{dt}m(t) \sim \frac{v}{s} = \rho_L d_L \text{ , initial condition : } m(0) = \frac{\rho_L}{r}$$

r : average ratio of occupied sites after nucleation.

$\Rightarrow$

$$m(t) = C_1 \rho_L d_L t + \frac{\rho_L}{r}$$

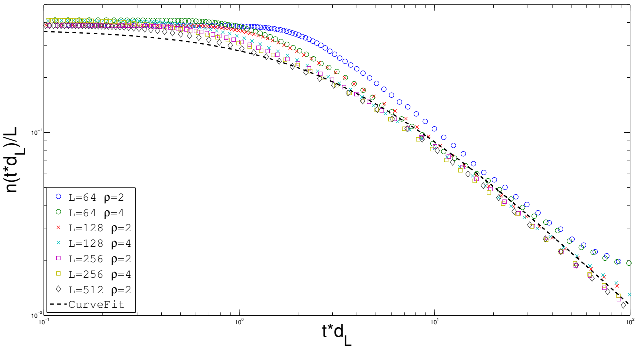
After time τ_L^{coars}, $m(t)$ will grow to size L.

$$m(\tau_L^{coars}) \sim L \;\Rightarrow\; \tau_L^{coars} \sim \frac{L}{d_L}$$

Coarsening Regime

Ratio of occupied sites:

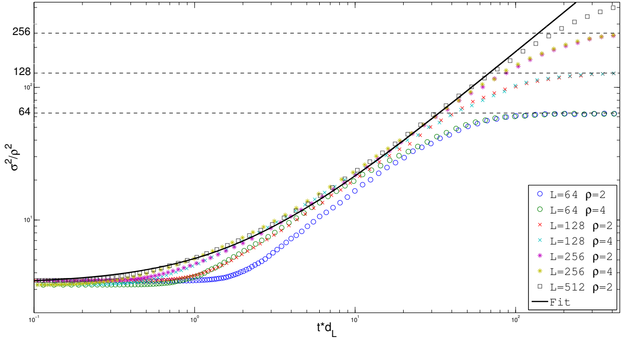
$$\frac{n(t)}{L} = \frac{1}{C_1 d_L t + 1/r} \ ,$$



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations. Fitting constant $C_1 \approx 0.8538$

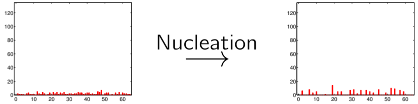
Second moment:

$$\sigma_L^2(t) \sim \frac{1}{L} n(t) m^2(t) \;\Rightarrow\; \frac{\sigma_L^2(t)}{\rho_L^2} = \tilde{C}\left(C_1 d_L t + \frac{1}{r}\right)$$



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations. Fitting constant $C_1 \approx 0.5513$ and $\tilde{C} \approx 3.3070$

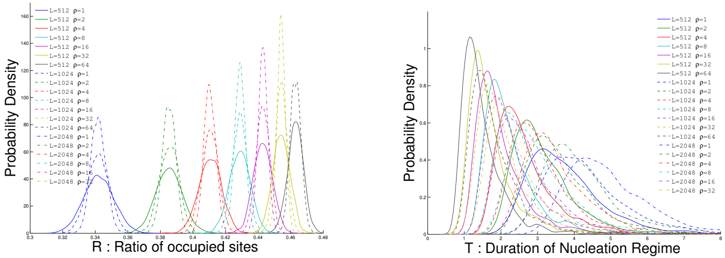
Nucleation Regime [in progress]



- Time scale of this regime is much smaller compared with coarsening regime.
- Dominated by **inclusion** $\Rightarrow$ very different dynamics.
- **Striped patterns**. Length grows exponentially with initial density

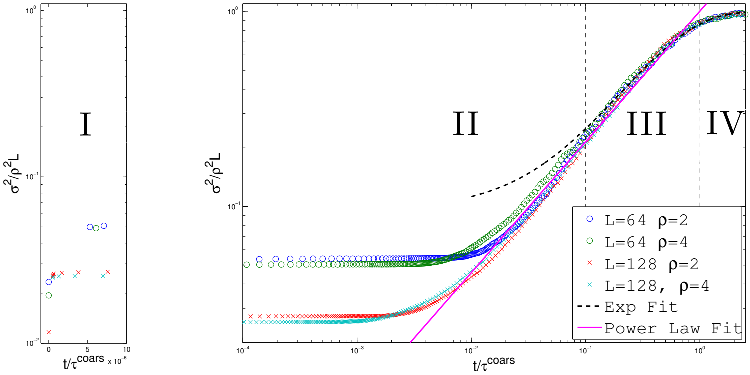
$$\mathbb{E}[l] = C_1 e^{C_2 \rho^{1/3}}$$

- **Ratio of occupied sites** $R \to 1/2$ as $\rho \to \infty$.



Probability density functions of occupied ratio and duration of nucleation regime for different system size and initial density. Data averaged on 2000 simulations. Pdf fitted with MATLAB.

- Ratio of occupied sites : approx. Normal distributed.
- Duration : approx. Gumbel distributed.



Simulation results with $d_L = 1/L^2$, averaged with 200 realizations

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THANK YOU