Interacting particles in open chaotic flows

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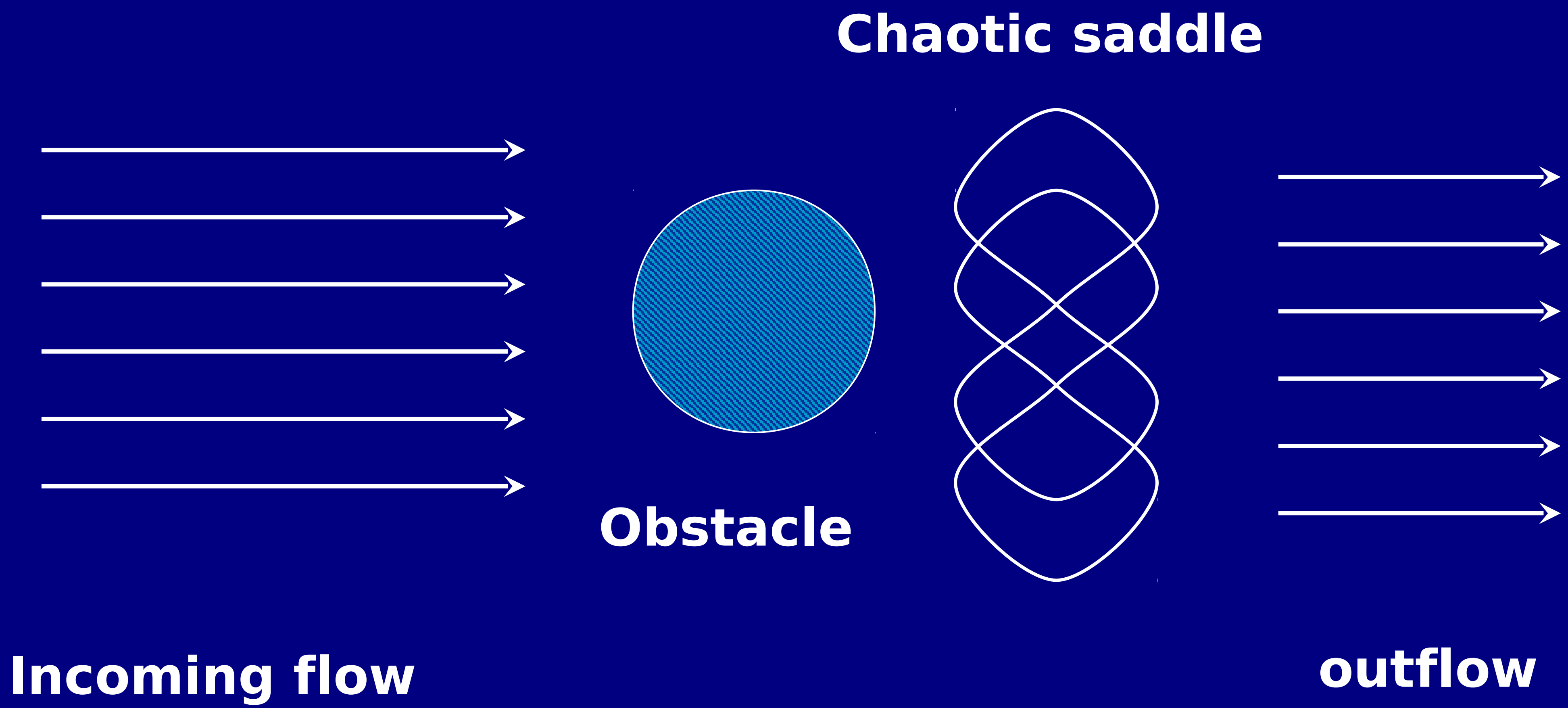
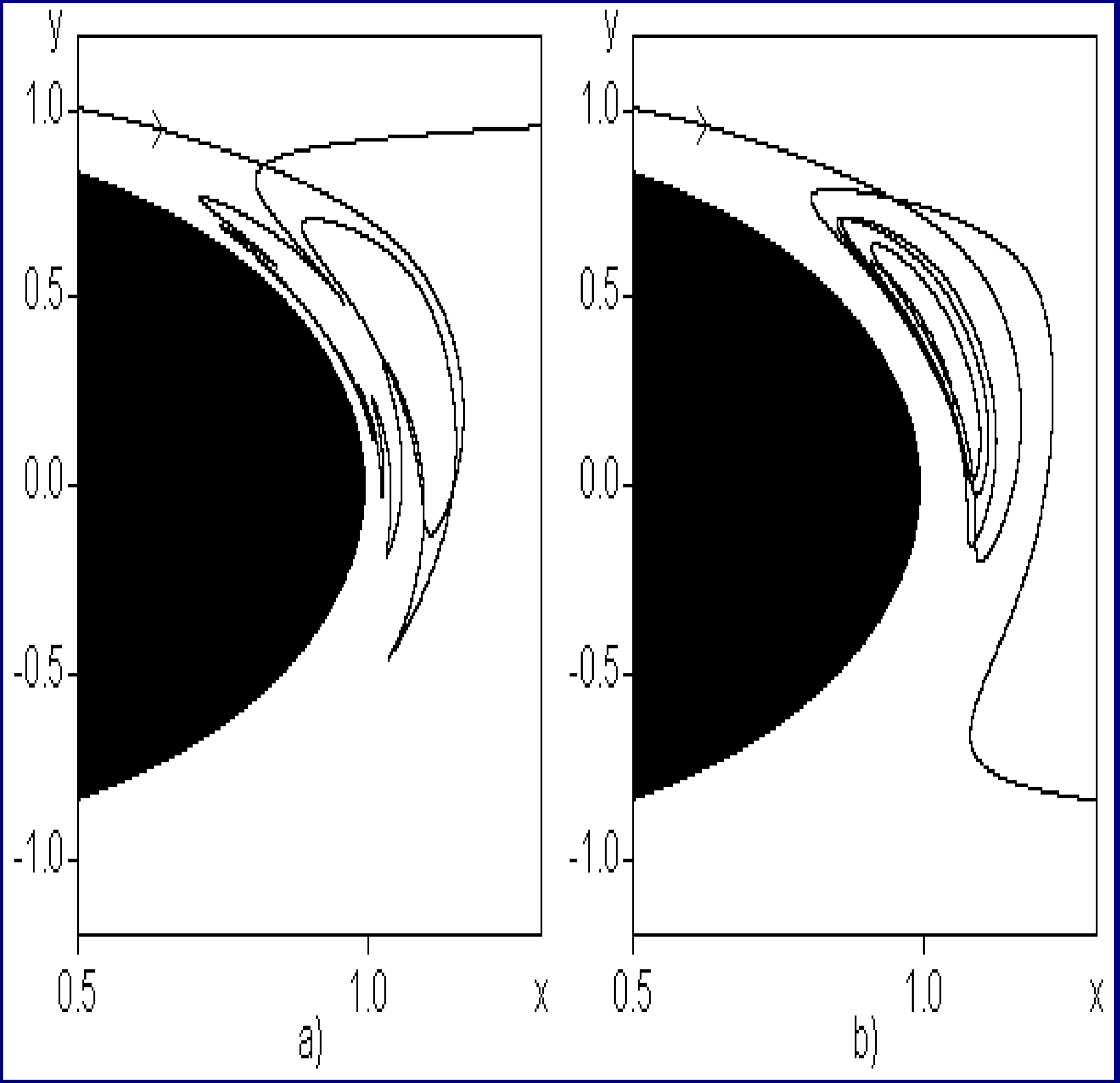
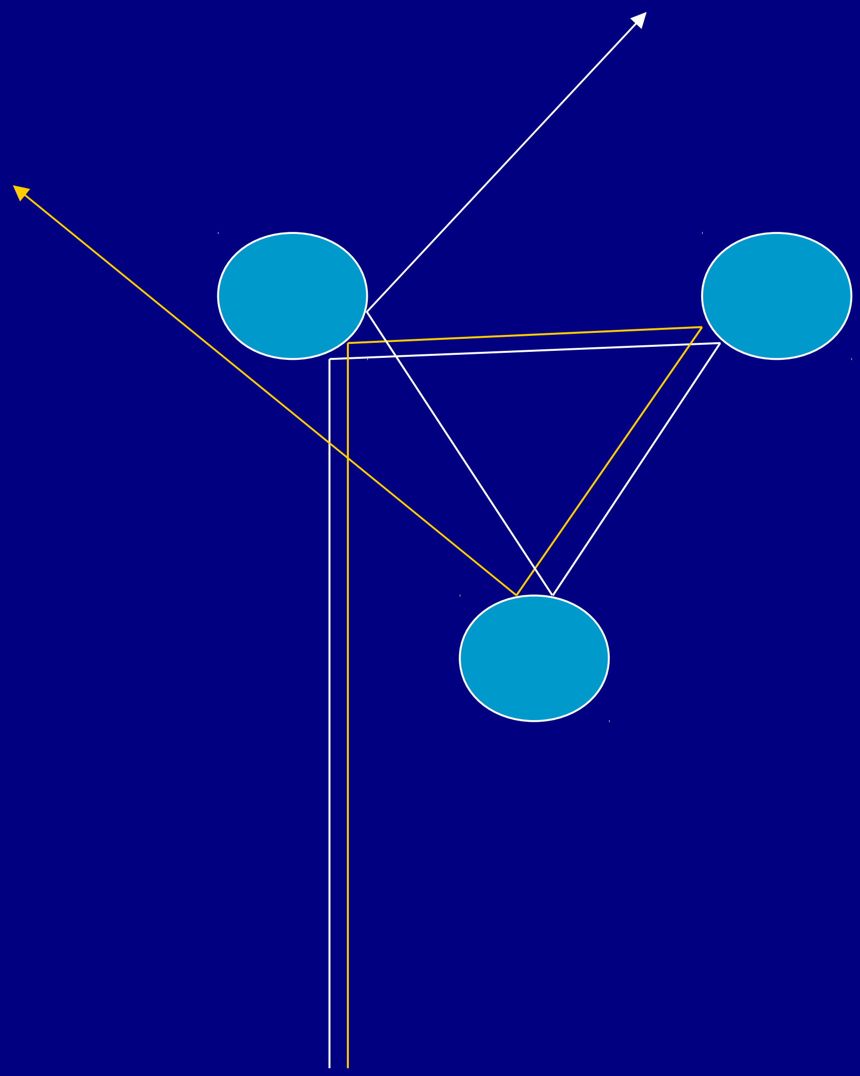
Open chaotic flows



Open flows can be chaotic
Example of the channel flow

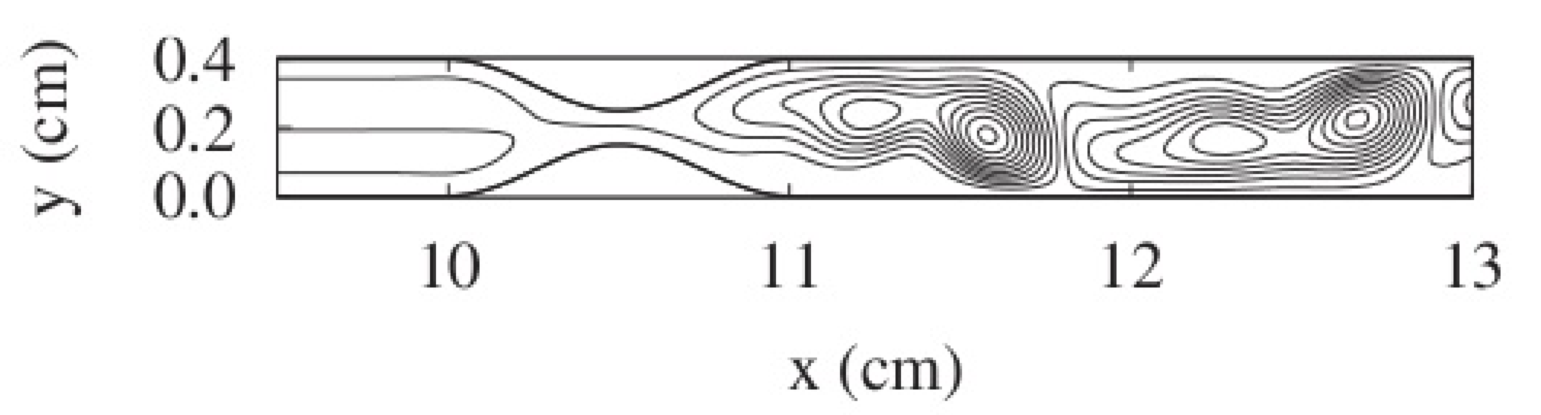


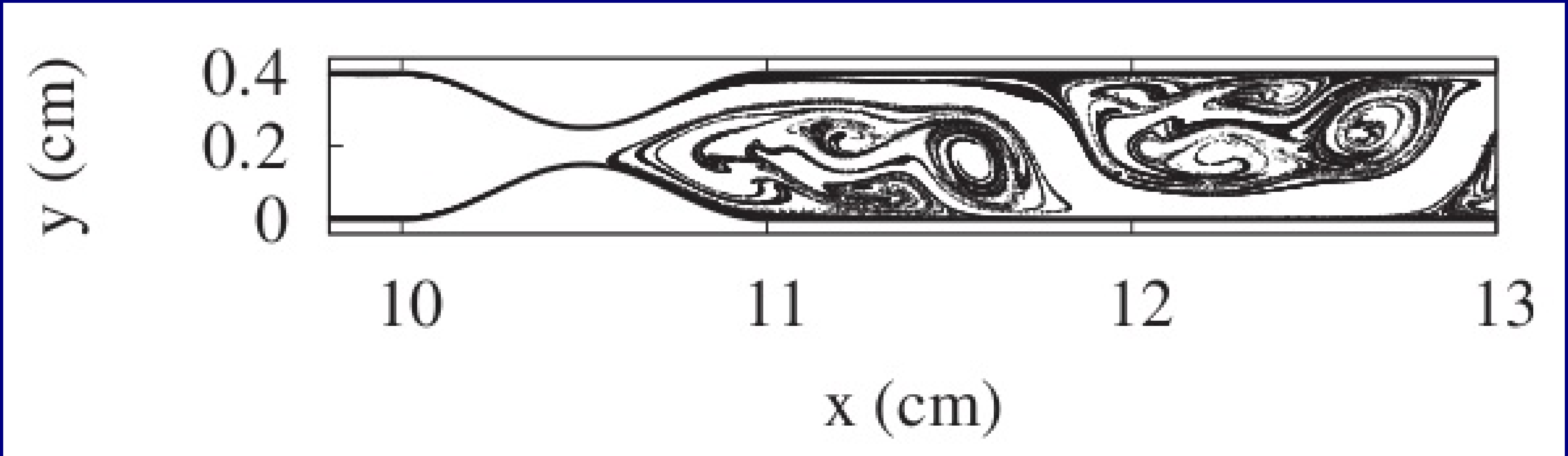
Advection in chaotic flows is an instance of chaotic scattering



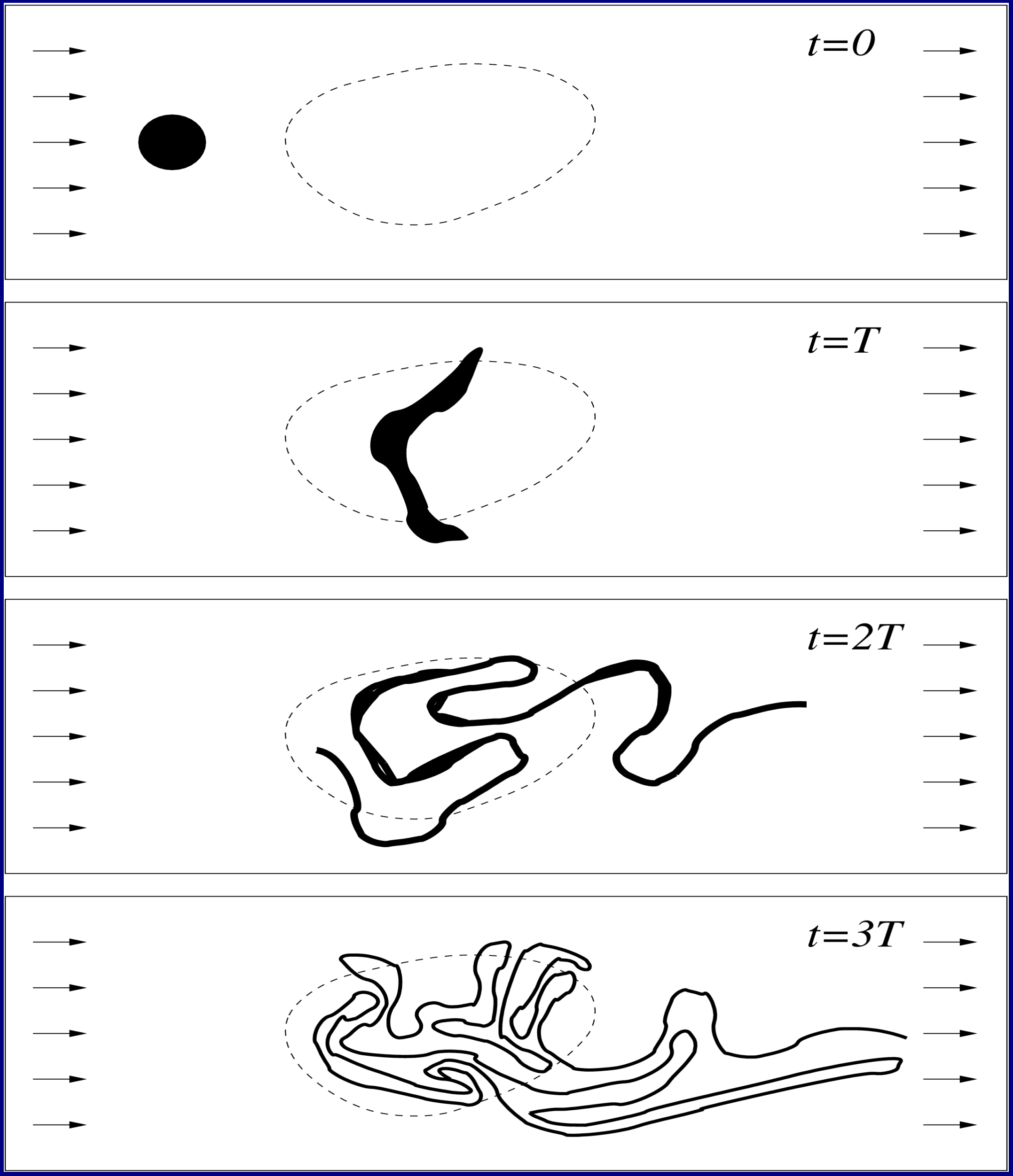
Example: blood flow

Scheling et al., 2009

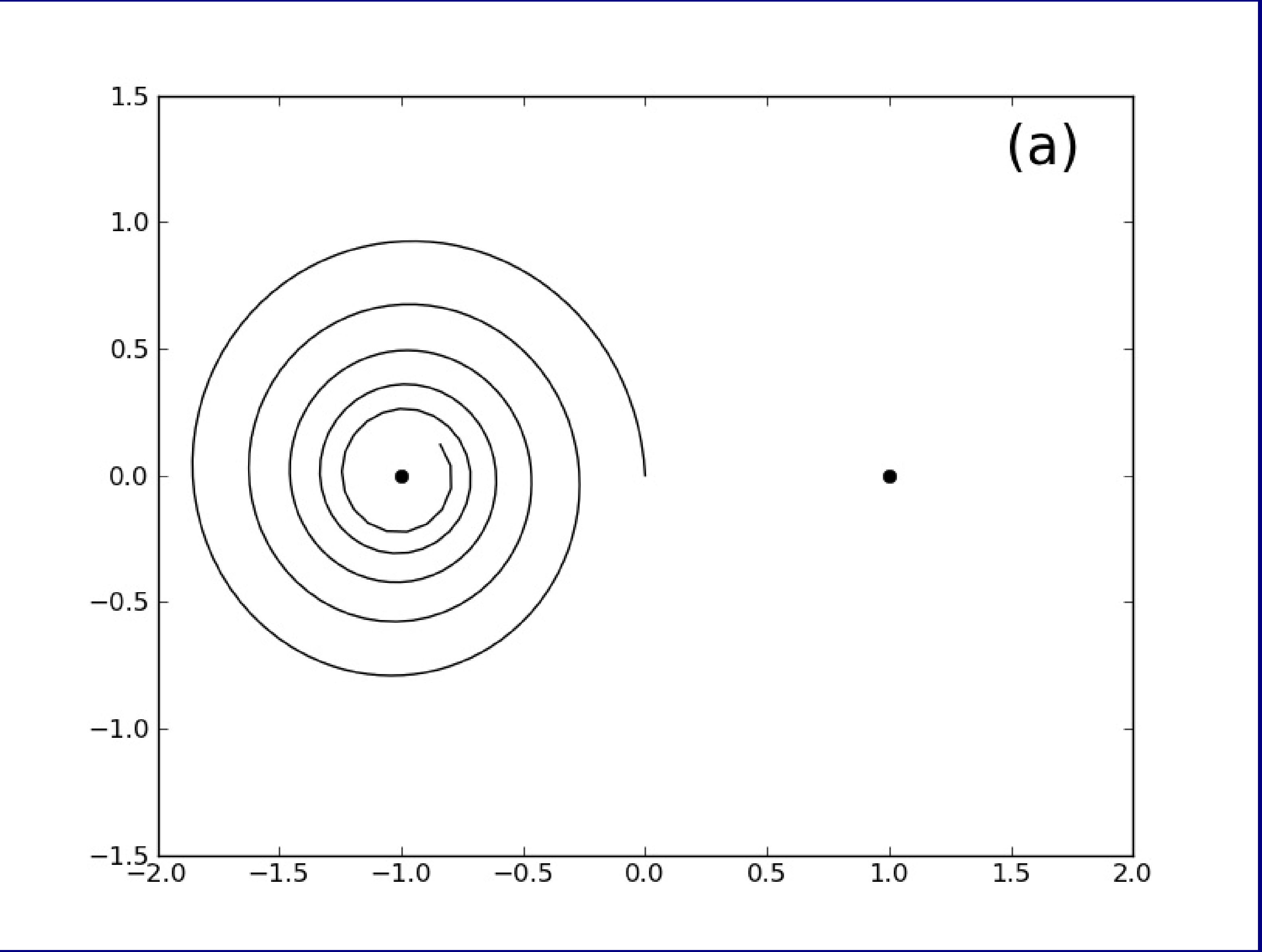




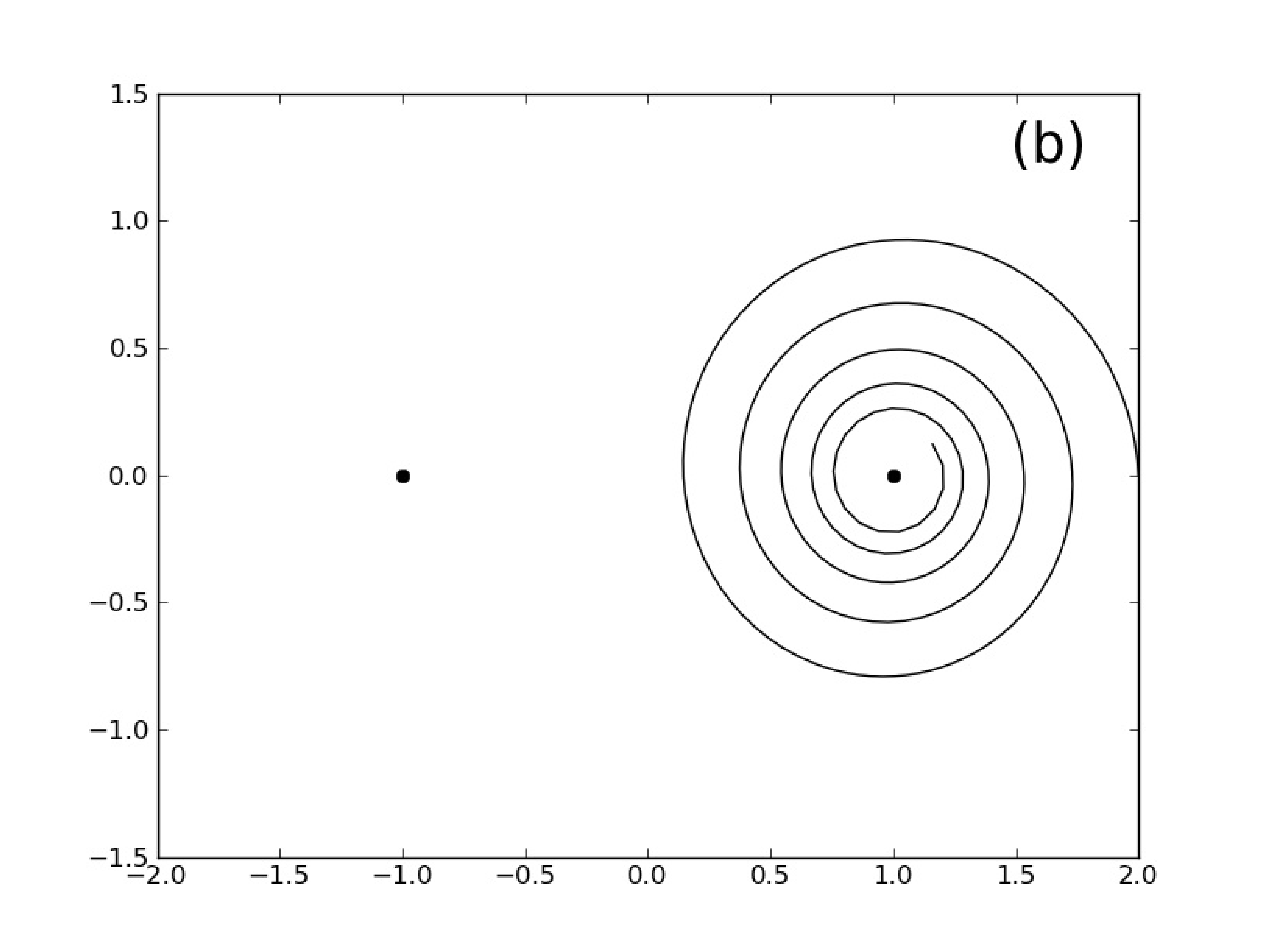
Chaotic open flows lead to the formation of fractal advective structures



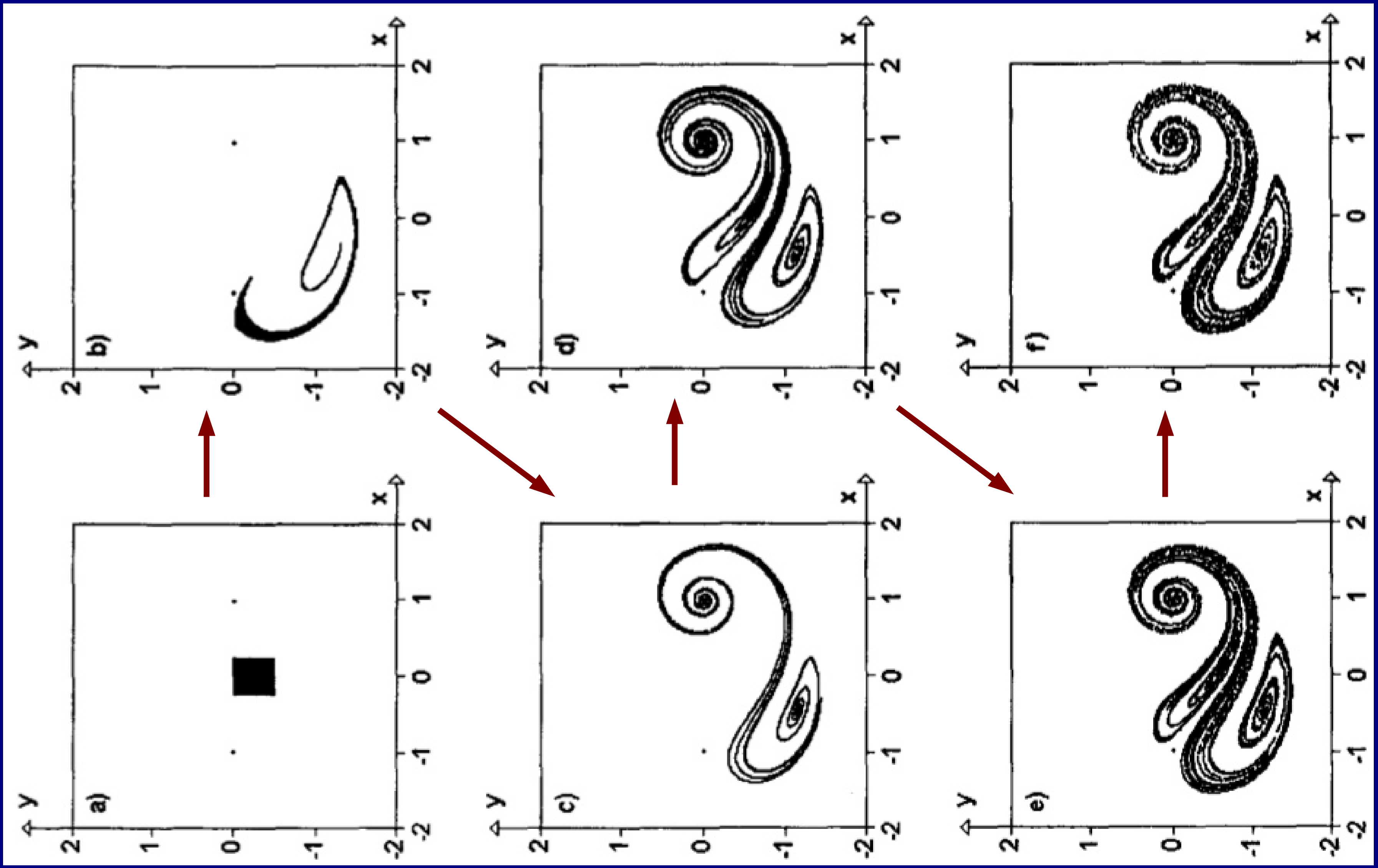
Example: blinking vortex-sink flow



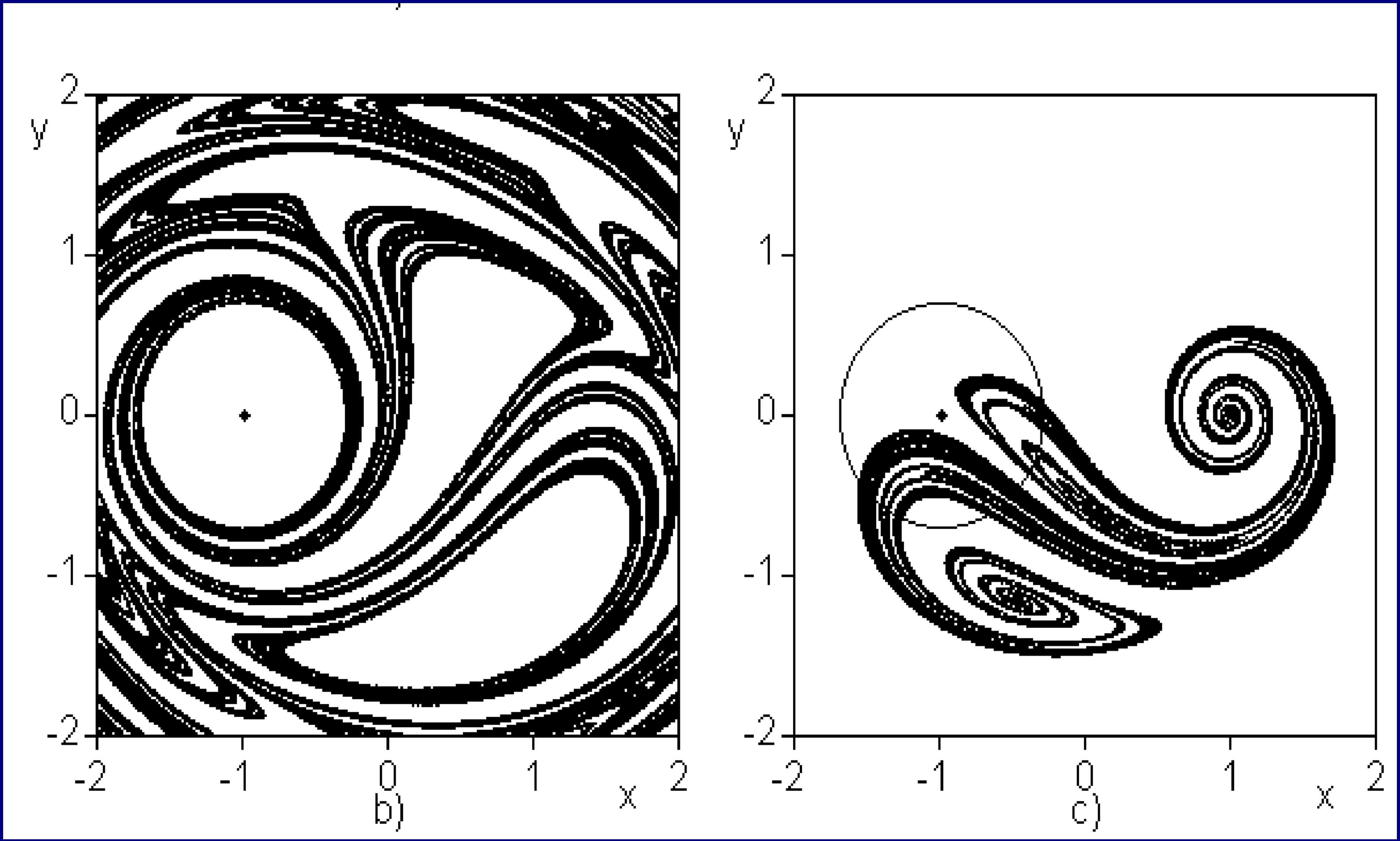
Example: blinking vortex-sink flow



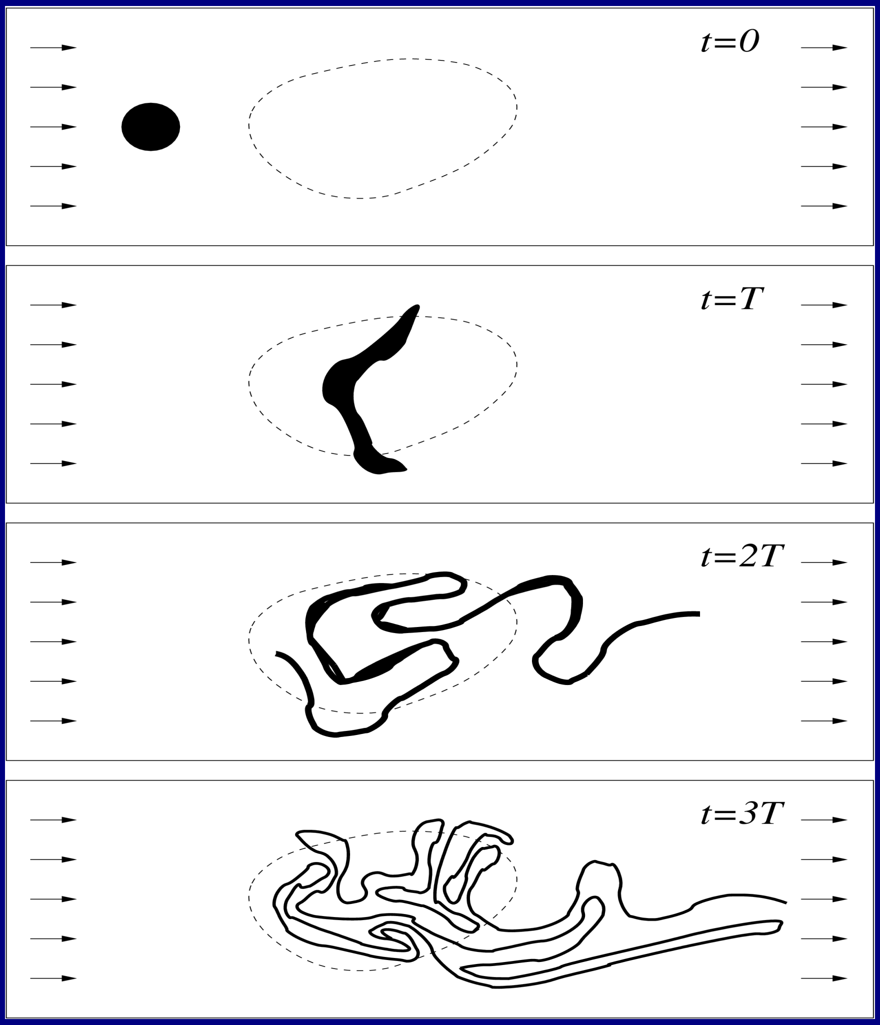
Evolution of a droplet of dye in the blinking vortex-sink system



Stable and unstable manifolds



Effects of fractal structures on chemical and biological processes



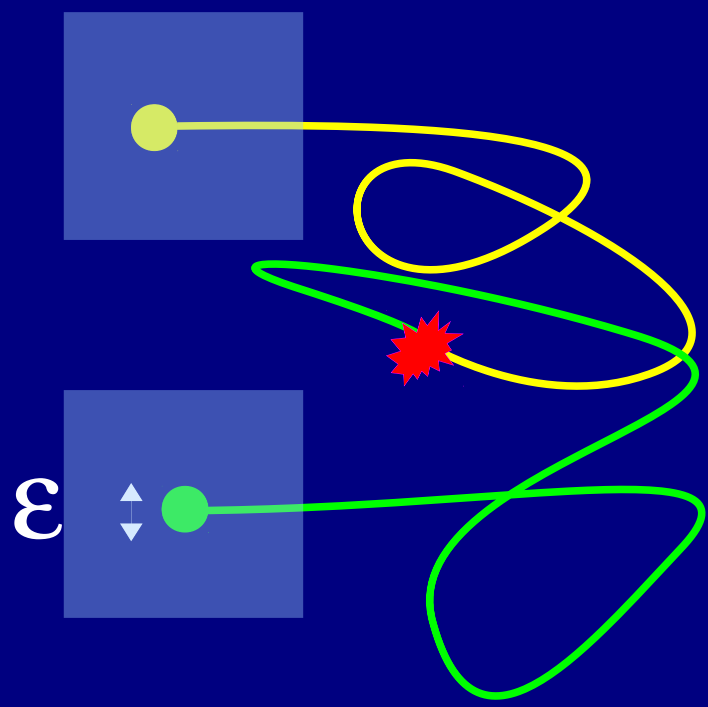
- Fractal distribution of advected material increases contact area (or perimeter) between reacting species.
- This results in a production term in the reaction rate equation which has anomalous scaling, of the form:

$$Q \sim B^{(1-D_1)/(2-D_1)}$$

But...

- This result is only valid in the low-density limit, when the mean free path of the reacting particles is much smaller than the characteristic length scales of the system.
- It is therefore worthwhile investigating the effect of chaotic advection and fractal structures on reactions from the kinetic theory point of view, taking into account collision events between pairs of particles.

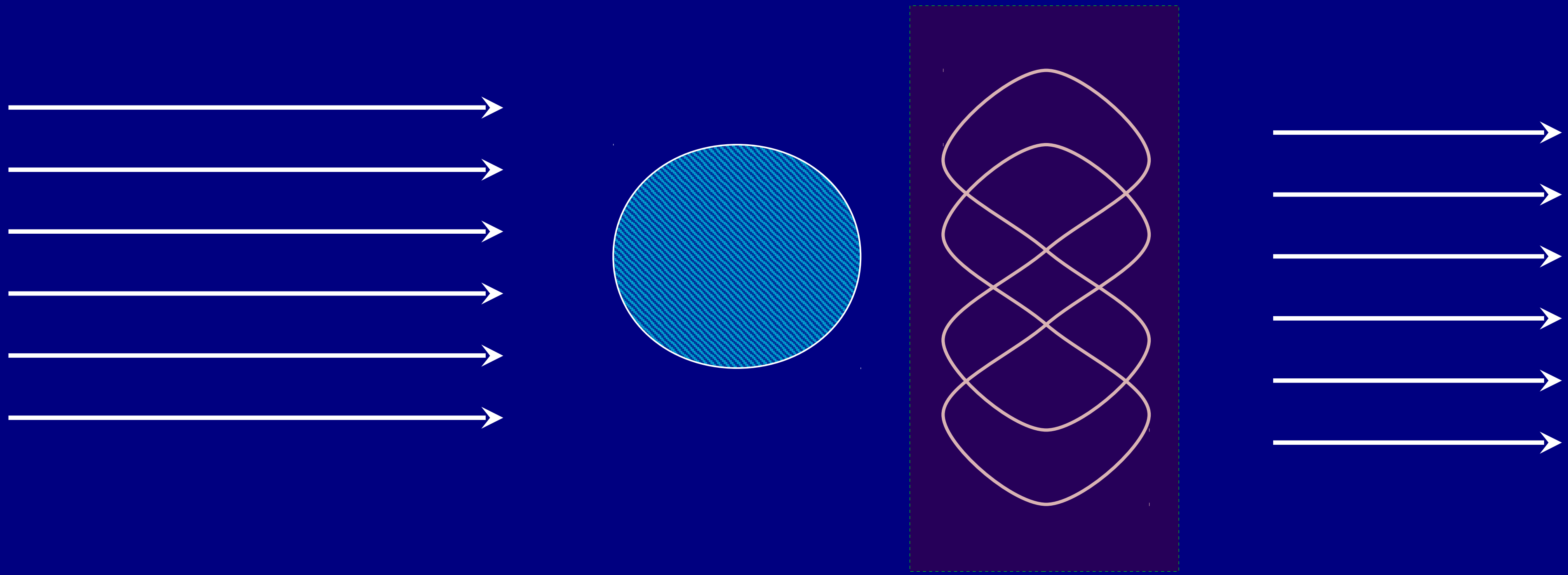
The collision probability and its scaling



- Reaction happens if particles come within a distance **ε** of each other.
- Let ***p(ε)*** be the probability that two particles with random initial conditions chosen in disjoint regions react before they escape.
- **Question:** how does ***p(ε)*** scale with ***ε***?

Smoluchovski model

Assume that the mixing region surrounding the chaotic saddle can be considered persfectly mixing during the residence time of the particles: the “chemical reactor approximation”

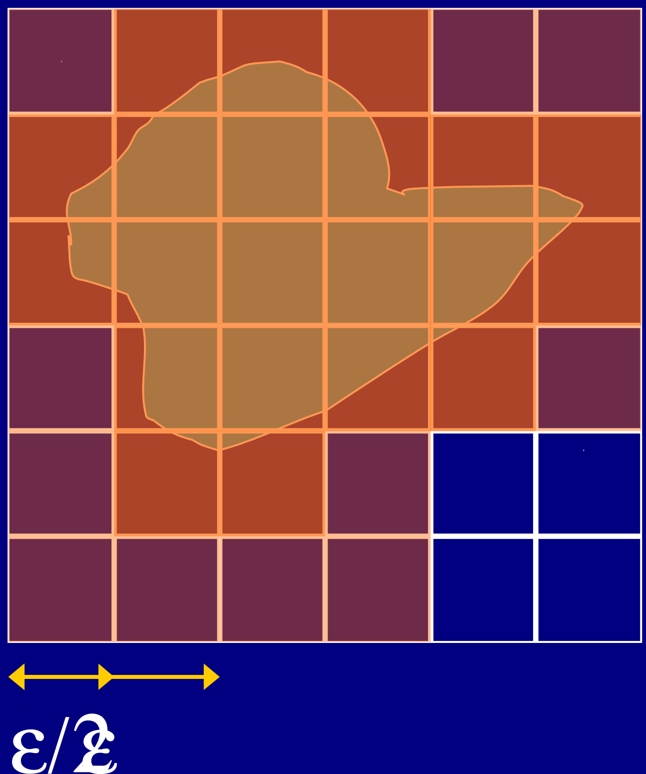


This leads to the prediction $p(\varepsilon) \sim \varepsilon^2$

But...

- The mixing caused by chaotic advection in open flows is **not** perfect or uniform.
- This derivation ignores the fractal structures generated by the chaotic motion.
- In particular, mixing is only achieved in a neighbourhood of size ε of the unstable manifold.
- Another issue is that not all particles will be mixed, but only those whose initial conditions lie in a neighbourhood of the stable manifold.

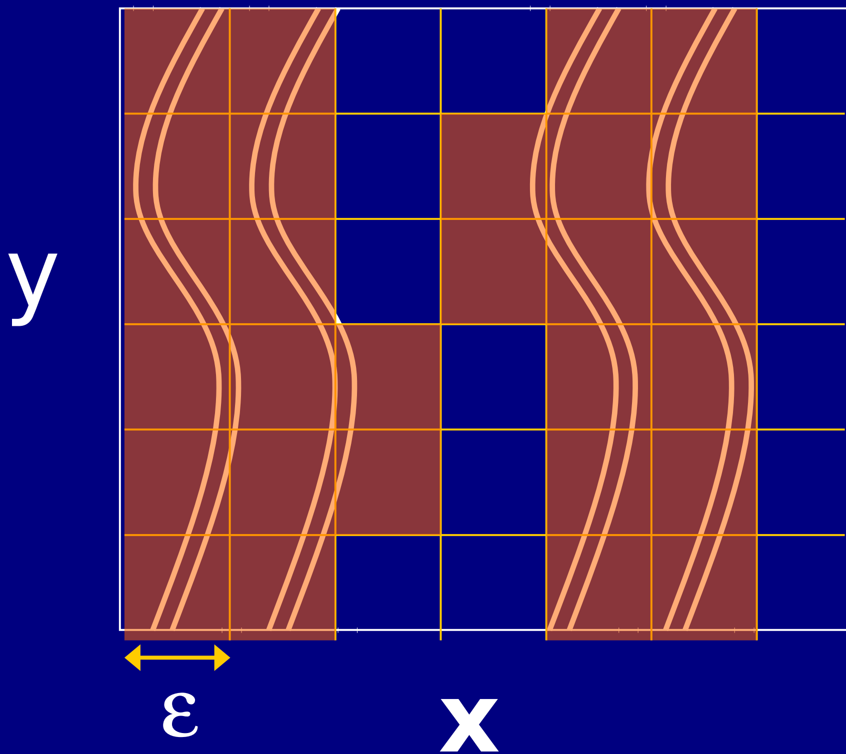
General definition of dimension



In an ambient space of dimension D_0, the number $N(\varepsilon)$ of "boxes" of size ε necessary to cover an object of dimension D scales as:

$$N(\varepsilon) \sim \varepsilon^{-D} \; ,$$

for $\varepsilon \to 0$. D is the **box-counting dimension**, or **fractal dimension**, and coincides with the intuitive idea of dimension for usual geometric shapes.



For a given resolution ε, the area of the "uncertain" region scales as:

$$A(\varepsilon) \sim \varepsilon^2 N(\varepsilon) \sim \varepsilon^{2-D}$$

This area is proportional to the probability of being within a distance ε of the boundary.

Expression for $p(\varepsilon)$

$$p(\varepsilon) \sim \varepsilon^{2-D_1} \bullet \varepsilon^{2-D_1} \bullet \varepsilon^{D_2}$$

Prob that particle 1
Is close enough to
The stable manifold

Prob that particle 2
Is close enough to
The stable manifold

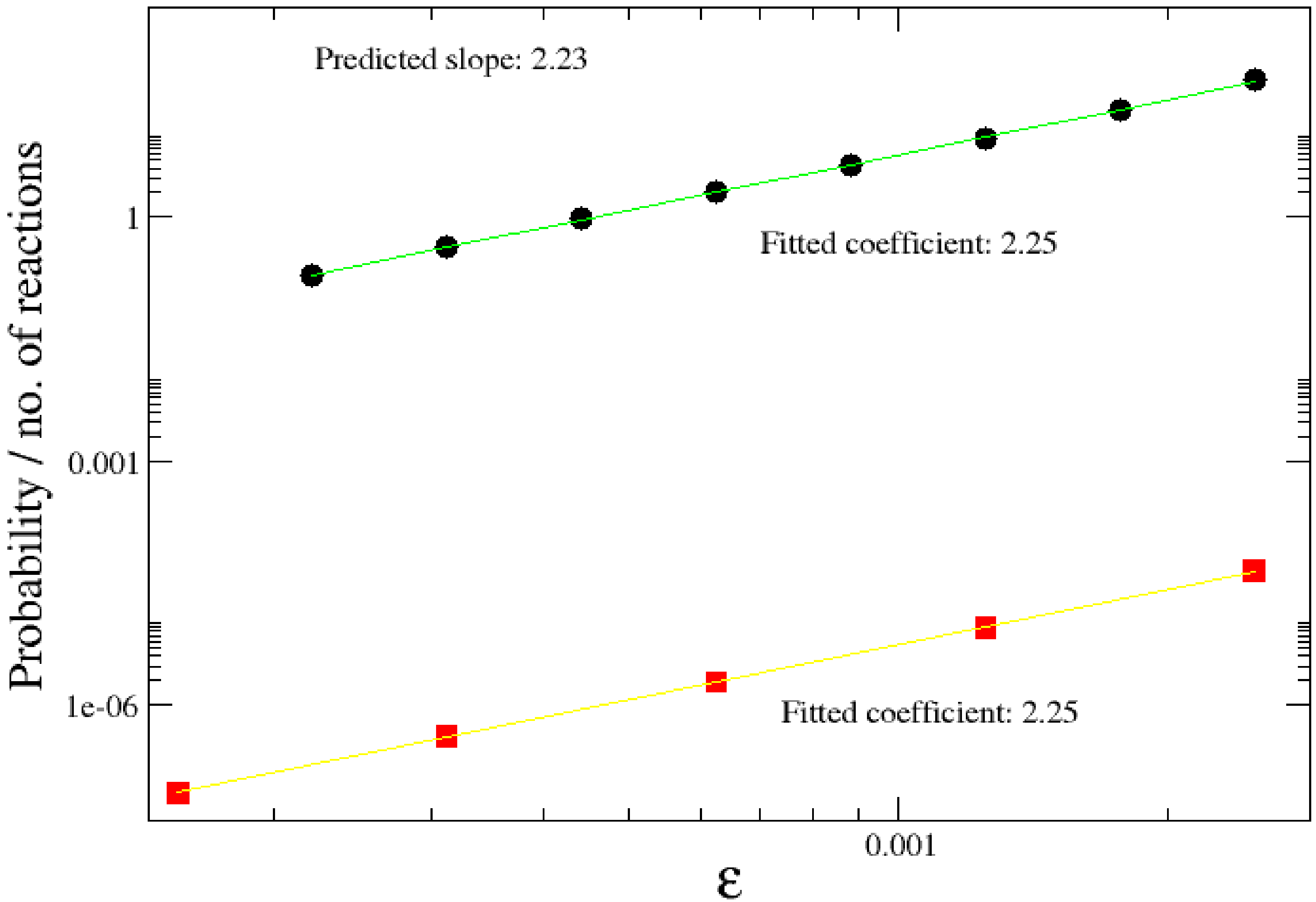
Prob that particles 1 and 2 meet each
other in the neighbourhood of the
Unstable manifold.

$$p(\varepsilon) \sim \varepsilon^{4-2D_1+D_2}$$

Numerical experiments

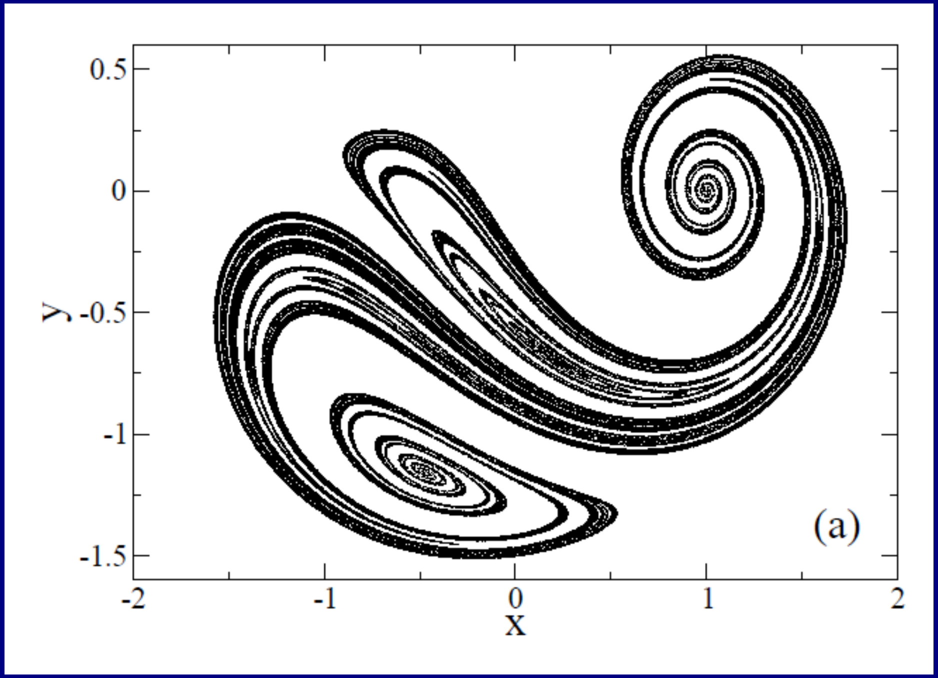
- $D_1 = 1.74$
- $D_2 = 1.71$
- This leads to a predicted scaling of $p(\varepsilon) \sim \varepsilon^{2.23}$

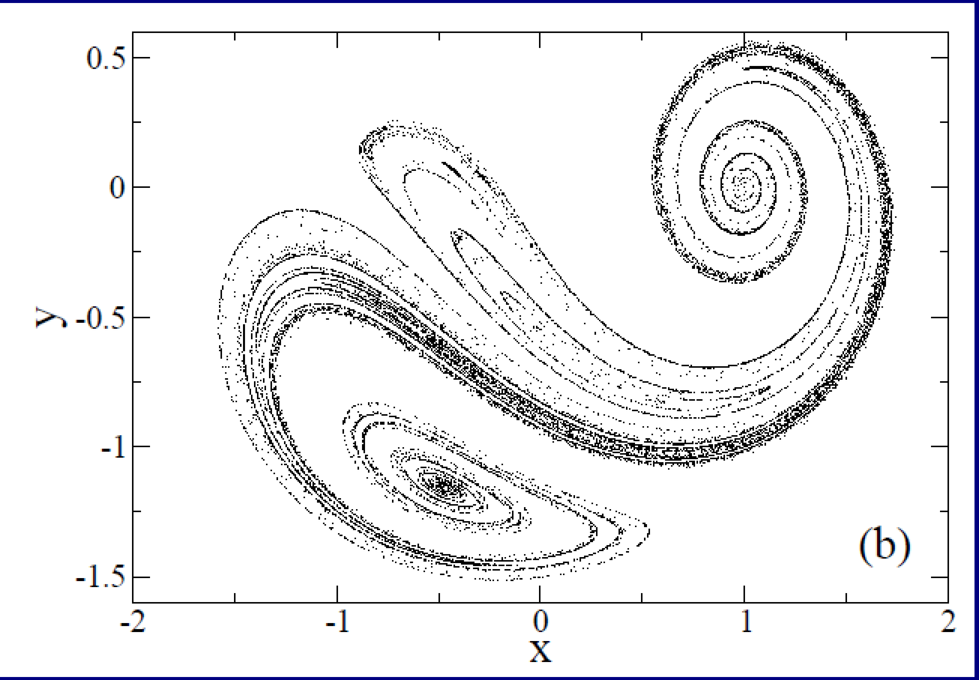
This same scaling is predicted for many particles, as long as their number is not too large. This scaling then governs the amount of reaction that takes place before the particles escape.



Predicted slope: 2.23
Fitted coefficient: 2.25
Fitted coefficient: 2.25
1
0.001
1e-06
Probability / no. of reactions
0.001
ε

Where the collisions take place





Unstable set

Locations of collisions

Conclusions

- The expression for the scaling of the collision probability was also verified in other systems displaying transient chaos (the open sympletic Henon map and an open variation of the Baker map), and the expression derived here agrees with the numerical results.
- This result is also valid for 3D hyperbolic open flows.
- Our derivation assumes the chaotic dynamics to be hyperbolic.

References

- Reviews: Tel, Moura, Grebogi, Karolyi, *Physics Reports* 2005; Moura, Gouillart, Feudel, *Advances in Applied Mechanics, 2012*.
- Chaotic scattering and transient chaos in general: Tel and Lai, *Transient chaos*, Springer, 2011.
- Collision results: Moura, *Physical Review Letters*, 2012.