$\ln e^L = L$ and FSS for 1st Order Phase Transitions

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Plan of talk

First and Second order phase transitions

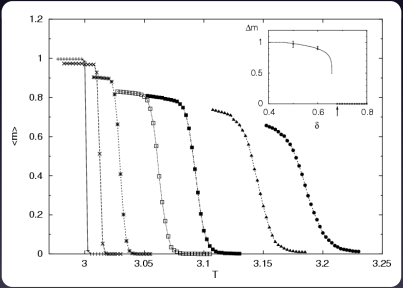
FSS at first order transitions, boring?

Non-standard FSS at first order transitions.....

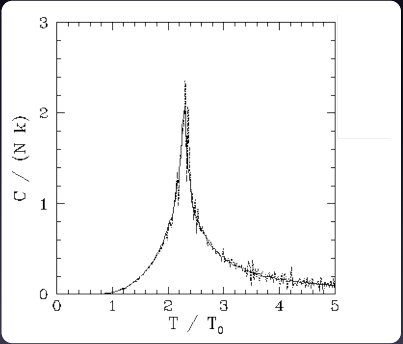
..... as a consequence of $\ln e^L = L$

Order of transition

First order - discontinuities in magnetization, energy (latent heat)



Second order - divergences in specific heat, susceptibility



The q-state Potts model

Hamiltonian

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$$

Evaluate a partition function

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta \mathcal{H}_q)$$

Extract physical quantities from derivatives of free energy

$$F = \ln Z$$

Potts - Order of transition?

Order of transition - dimension, symmetry

$$\mathcal{H}_q = -\sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j}$$

Two dimensions ferromagnetic, $q = 2, 3, 4$ continuous

$d \geq 3$ (and mean field), $q = 2$ continuous otherwise first order

Heuristic two-phase model

A fraction W_o in q ordered phase(s), energy $\hat{e}_\mathrm{o}$

A fraction $W_\mathrm{d} = 1 - W_\mathrm{o}$ in disordered phase, energy $\hat{e}_\mathrm{d}$

The hat $\implies$ quantities evaluated at β^∞.

Neglect fluctuations *within* the phases

Energy moments

Energy moments become

$$\langle e^n \rangle = W_{\mathrm{o}} \hat{e}_{\mathrm{o}}^n + (1 - W_{\mathrm{o}}) \hat{e}_{\mathrm{d}}^n$$

And the specific heat then reads:

$$C_V(\beta, L) = L^d \beta^2 \left(\langle e^2 \rangle - \langle e \rangle^2 \right) = L^d \beta^2 W_{\mathrm{o}} (1 - W_{\mathrm{o}}) \Delta \hat{e}^2$$

Max of $C_V^{\max} = L^d \, (\beta^\infty \Delta \hat{e}/2)^2$ at $W_{\mathrm{o}} = W_{\mathrm{d}} = 0.5$

Volume scaling

FSS: Specific Heat

Probability of being in any of the states

$$p_\mathrm{o} \propto e^{-\beta L^d \hat{f}_\mathrm{o}} \text{ and } p_\mathrm{d} \propto e^{-\beta L^d \hat{f}_\mathrm{d}}$$

Time spent in the ordered states $\propto q p_\mathrm{o}$

$$W_\mathrm{o}/W_\mathrm{d} \simeq q e^{-L^d \beta \hat{f}_\mathrm{o}} / e^{-\beta L^d \hat{f}_\mathrm{d}}$$

Expand around β^∞

$$0 = \ln q + L^d \Delta\hat{e}(\beta - \beta^\infty) + \dots$$

Solve for specific heat peak

$$\beta^{C_V^\mathrm{max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta\hat{e}} + \dots$$

FSS: Binder Cumulant

Location of min $\beta^{B^{\min}}(L)$

$$\beta^{B^{\min}}(L) = \beta^{\infty} - \frac{\ln(q\hat{e}_{\mathrm{o}}^2/\hat{e}_{\mathrm{d}}^2)}{L^d \Delta\hat{e}} + \ldots$$

Energetic Binder cumulant

$$B(\beta, L) = 1 - \frac{\langle e^4 \rangle}{3\langle e^2 \rangle^2}\ .$$

$\frac{1}{L^d}$ FSS

Exponential Degeneracy

Normally q is constant

Suppose instead $q \propto e^L$

$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{\ln q}{L^d \Delta\hat{e}} + \ldots$$

$$\beta^{B^{\min}}(L) = \beta^\infty - \frac{\ln(q\hat{e}_{\mathrm{o}}^2/\hat{e}_{\mathrm{d}}^2)}{L^d \Delta\hat{e}} + \ldots$$

become

$$\beta^{C_V^{\max}}(L) = \beta^\infty - \frac{1}{L^{d-1} \Delta\hat{e}} + \ldots$$

$$\beta^{B^{\min}}(L) = \beta^\infty - \frac{\ln(\hat{e}_{\mathrm{o}}^2/\hat{e}_{\mathrm{d}}^2)}{L^{d-1} \Delta\hat{e}} + \ldots$$

Exponential Degeneracy: summary

If $q \propto e^L$ ....

We see $\frac{1}{L^{d-1}}$ finite size scaling, *NOT* $\frac{1}{L^d}$

Examples?

Exponential degeneracy and 1st order transition

A 3D Plaquette Ising action

$3D$ cubic, spins on *vertices*

$$H = -\frac{1}{2} \sum_{[i,j,k,l]} \sigma_i \sigma_j \sigma_k \sigma_l$$

NOT

$$H = -\sum_{[i,j,k,l]} U_{ij} U_{jk} U_{kl} U_{li} \quad , \quad U_{ij} = \pm 1 \qquad \mathcal{Z}_2 \;\; Lattice\ Gauge$$

And the dual....

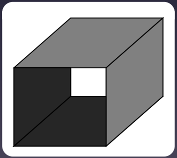
Dual, by hand

$$Z(\beta) = \sum_{\{\sigma\}} \prod_{[ijkl]} \cosh(\beta) \left[1 + \tanh(\beta)\left(\sigma_i \sigma_j \sigma_k \sigma_l\right)\right]$$

which can be written as

$$Z(\beta) = \left[2\cosh(\beta)\right]^{3L^3} \sum_{\{S\}} \left[\tanh(\beta)\right]^{n(S)}$$

“Matchbox” spins

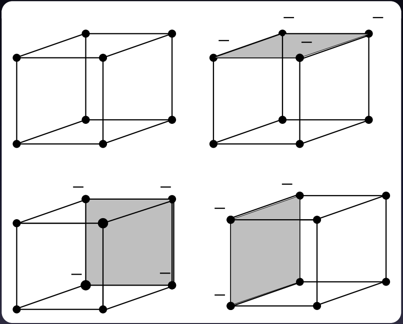


And the dual.... II

An anisotropically coupled Ashkin-Teller model

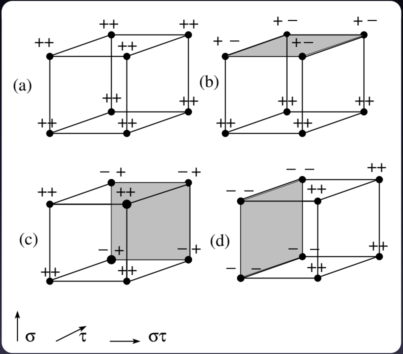
$$H_{dual} = -\frac{1}{2}\sum_{\langle ij\rangle_x}\sigma_i\sigma_j - \frac{1}{2}\sum_{\langle ij\rangle_y}\tau_i\tau_j - \frac{1}{2}\sum_{\langle ij\rangle_z}\sigma_i\sigma_j\tau_i\tau_j\ ,$$

Groundstates: Plaquette



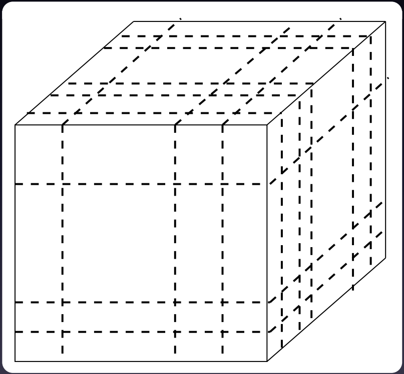
Persists into low temperature phase: degeneracy 2^{3L}

Groundstates: Dual



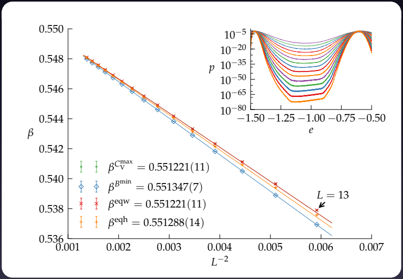
Dual degeneracy

Ground state

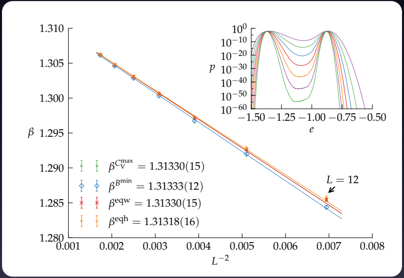


FSS

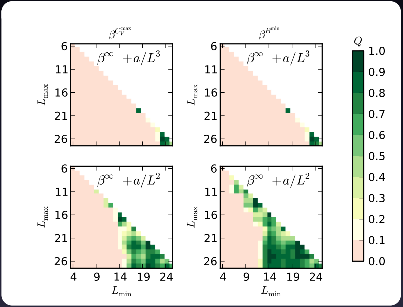
Plaquette Hamiltonian fits



Dual Hamiltonian fits



Quality of fits



Forcing a fit to $1/L^3$ gives much poorer quality

Conclusions

Standard 1st order FSS: $1/L^3$ corrections in 3D

Fixed BC: $1/L$ (surface tension)

Exponential degeneracy: $1/L^2$ in 3D

Further applications may be higher-dimensional variants of the gonihedric model, ANNNI models, spin ice systems,"orbital" compass models, . . .

References

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