Extreme Value Statistics of 1/f Noises

Statistical Mechanics Approach

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References:

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Summary of standard extreme-value statistics:

• Let $z_1, \ldots, z_N$ be i.i.d. random variables with probability density function $p(z)$. Let $y_N = max_{i=1,\ldots,N}\{z_n\}$ be **maximum** of the set, and $F_N(y) = Prob(y_N < y)$ be the **distribution of the maximum**. Then for $N \gg 1$ the distribution approaches a **scaling form** $F_N(y) \approx F_\infty\left[(y + a_N)/b_N\right]$ where a_N, b_N depend on $p(z)$ but the shape of F_∞ is **universal**, and given by

$$F_\infty(y) = \begin{cases} e^{-e^{-y}}, \quad \forall y & \textbf{Gumbel class: } z < \infty \text{ and } p(z \to \infty) \sim Ae^{-z^\alpha},\ \alpha > 0 \\ e^{-y^{-\alpha}}, \ y \geq 0 & \textbf{Fréchet class: } z < \infty \text{ and } p(z \to \infty) \sim Az^{-(\alpha+1)} \\ e^{-[-y]^\alpha}, \ y \leq 0 & \textbf{Weibull class: } z < a \text{ and } p(z \to a) \sim A(a-z)^{(\alpha-1)} \end{cases}$$

The result is rather robust if variables are short-range correlated. In particular, for Gaussian-distributed variables with $< z_i >= 0$ the **Gumbel** distribution is known to be valid as long as $C(|i-j|) = \langle z_i z_j \rangle \lesssim const/\ln|i-j|$ for $|i-j| \gg 1$.

Very few explicit results exist for extrema of **strongly correlated variables**, as e.g. for Brownian motion by **Majumdar & Comtet**, or for the largest eigenvalues of random matrices by **Tracy & Widom**.

Definition of $1/f$ noise:

Random signals such that **spectral power** associated with a given Fourier harmonic is **inversely proportional** to the frequency. Believed to be **ubiquitous** in Nature.

- **Periodic version:** random (Gaussian) Fourier series of the form

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right]$$

where $v_n, \overline{v}_n$ are **complex i.i.d.** with zero mean and the variance $\langle v_n \overline{v}_n \rangle = 1$. It implies

$$\langle V(t_1) V(t_2) \rangle_V = -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t \neq t' \in [0, 2\pi)$$

- Similarly, we can define the process on the whole line $-\infty < t < \infty$ by random Fourier integral

$$V(t) = \int_0^{\infty} \frac{d\omega}{\sqrt{\omega}} \left[e^{i\omega t} v(\omega) + e^{-i\omega t} \overline{v}_n(\omega) \right], \quad \langle V(t_1) V(t_2) \rangle_V = -2 \ln |t_1 - t_2|,$$

with $\delta-$correlated complex Gaussian $v(\omega)$. The corresponding definitions are formal, as sums/integrals do not converge pointwise, and should be understood after a proper **regularization**.

Regularization:

Subdivide the interval $t \in (0, 2\pi]$ by **finite number** of points $t_k = \frac{2\pi}{M} k$ where $k = 1, \ldots, M < \infty$ and associate with each k normally distributed real variables V_k with covariances

$$\langle V_k V_m \rangle = -2 \ln |2 \sin \frac{t_k - t_m}{2}|, \quad \text{for} \quad k \neq m$$

For the problem to be well-defined we have to choose the variance accordingly:

$$\langle V_k^2 \rangle = 2 \ln M + W, \quad \text{with any} \quad W > 0$$

In the limit $M \to \infty$ this is expected to approximate the $2\pi-$ periodic **1/f noise** with the two-point correlation $\langle V(t_1) V(t_2) \rangle_V = -2 \ln |2 \sin \frac{t_1 - t_2}{2}|, \quad t \neq t' \in [0, 2\pi)$.

- Our aim is to understand the statistics of **minima/maxima** of this **strongly correlated** sequence. The problem turns out to be intimately connected to the mechanism of **freezing transitions** in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to **Liouville Quantum Gravity** $\&$ conformal field theory, to **multifractal** random measures and processes arising in turbulence and mathematical finance, as well as to various aspects of the **Random Matrix Theory**.

Idea of the method:

We interpret the sequence V_k for $k = 1, \ldots, M < \infty$ as a set of **random energies** and consider the associated equilibrium Statistical Mechanics by introducing the temperature $T = \beta^{-1}$ and defining the **partition function**

$$Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$$

In this way we arrive to **1D generalization** of the **Derrida**'s Random Energy Model to be studied in the thermodynamic limit $M \to \infty$. The **minimal energy** can be extracted from the zero-temperature limit of the free energy as

$$V_{min} = \min(V_1, \ldots, V_M) = -\lim_{\beta \to \infty} \beta^{-1} \log Z(\beta)$$

Conclusion: We need to know the statistics of the **free energy** to extract the **extreme value statistics** of the energy sequence.

Observation: The positive integer moments $\langle Z^n(\beta)\rangle$, $n = 1, 2, \ldots$ of the partition function $Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$ for the (regularized) periodic logarithmic model in the high-temperature phase $\gamma = \beta^2 < 1$ turn out to be given in the thermodynamic limit $M \gg 1$ by

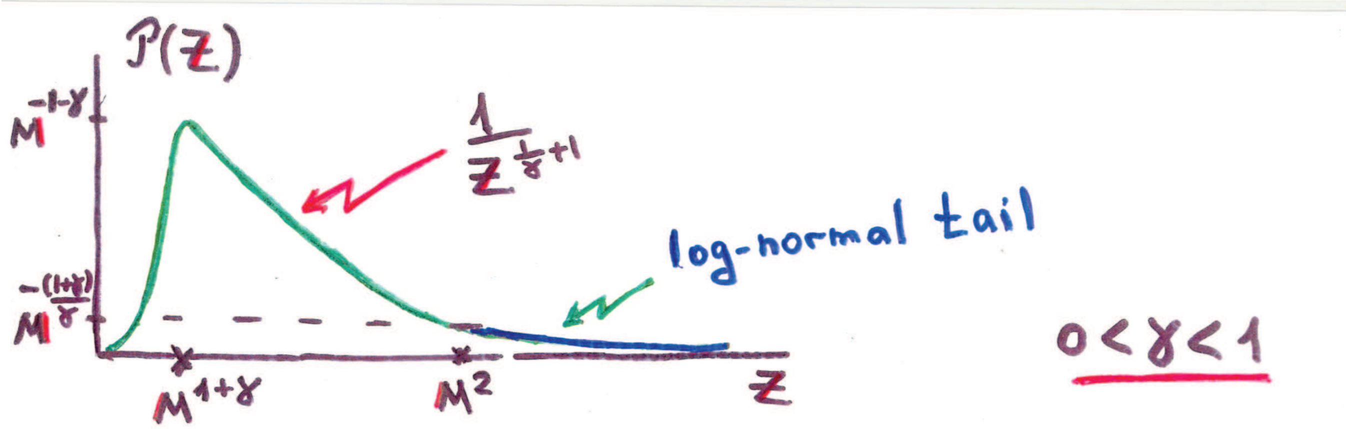
$$\langle Z^n(\beta)\rangle = \begin{cases} M^{1+\gamma n^2} O(1) & n > 1/\gamma \\ M^{(1+\gamma)n} D_n(\gamma) & n < 1/\gamma \end{cases}$$

where $D_n(\gamma)$ is the **Dyson-Morris** Integral

$$D_n(\gamma) = \frac{1}{(2\pi)^n}\int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_n \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^{-2\gamma} = \frac{\Gamma(1-n\gamma)}{\Gamma^n(1-\gamma)}$$

Aim: to reconstruct the distribution of the partition function $P(Z)$ from its moments in the high temperature phase $\gamma \leq 1$.

Outcome of the analysis: The probability density $\mathcal{P}(Z)$ of the partition function $Z_(\beta)$ in the high-temperature phase $\gamma = \beta^2 < 1$ consists of two pieces:



The **"body"** of the distribution has a pronounced maximum at $Z \sim Z_e = \frac{M^{1+\gamma}}{\Gamma(1-\gamma)} \ll M^2$, and the powerlaw decay at $Z_e \ll Z \ll M^2$:

$$\mathcal{P}(Z) = \frac{1}{\gamma}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}}}, \quad Z \ll M^2$$

At $Z \gg M^2$ it is replaced by the **lognormal tail.** Now we define $z = Z/Z_e$ and consider the generating function

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$$

Freezing scenario: In the high-temperature phase $\beta < \beta_c = 1$ the generating function $g_\beta(x)$ can be found explicitly and turned out to satisfy a remarkable **duality relation**:

$$g_\beta(x) = \int_0^\infty dt \exp\left\{-t - e^{\beta x} t^{-\beta^2}\right\}, \Rightarrow g_\beta(x) = g_{\frac{1}{\beta}}(x) \, .$$

This however does not allow to continue to $\beta > \beta_c$ regime. The phase transition at $\beta = \beta_c$ is believed to be described by the following **freezing scenario**: $g_\beta(x)$ **freezes** to the **temperature independent** profile $g_{\beta_c}(x)$ in the "glassy" phase $T \leq T_c$. The scenario is supported by
(i) a heuristic **real-space renormalization group arguments** for the logarithmic models (**Carpentier, Le Doussal '01**) revealing an analogy to the **travelling wave** analysis of polymers on disordered trees (**Derrida, Spohn 1989**)
(ii) **duality** which implies

$$\partial_\beta g_\beta(x)|_{\beta=\beta_c^-} = 0 \quad , \text{ for all } x$$

showing that the "temperature flow" of this function vanishes at the critical point $\beta = \beta_c = 1$
(iii) our **numerics**.

Assuming validity of such scenario for the problem in hand, one finds the frozen profile for the periodic model:

$$g_{\beta_c}(x) = 2e^{x/2}K_1(2e^{x/2})$$

where $K_1(z)$ is the Macdonald function. This allows to reconstruct the **distribution of the free energy** $f = -\beta^{-1}\ln z$ for any $T < T_c$. The corresponding formula takes a form of an infinite series:

$$\mathcal{P}^{CLM}_{\beta>\beta_c}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isf}\,\frac{1}{\Gamma(1+\frac{is}{\beta})}\Gamma^2\left(1+\frac{is}{\beta_c}\right)\,ds$$

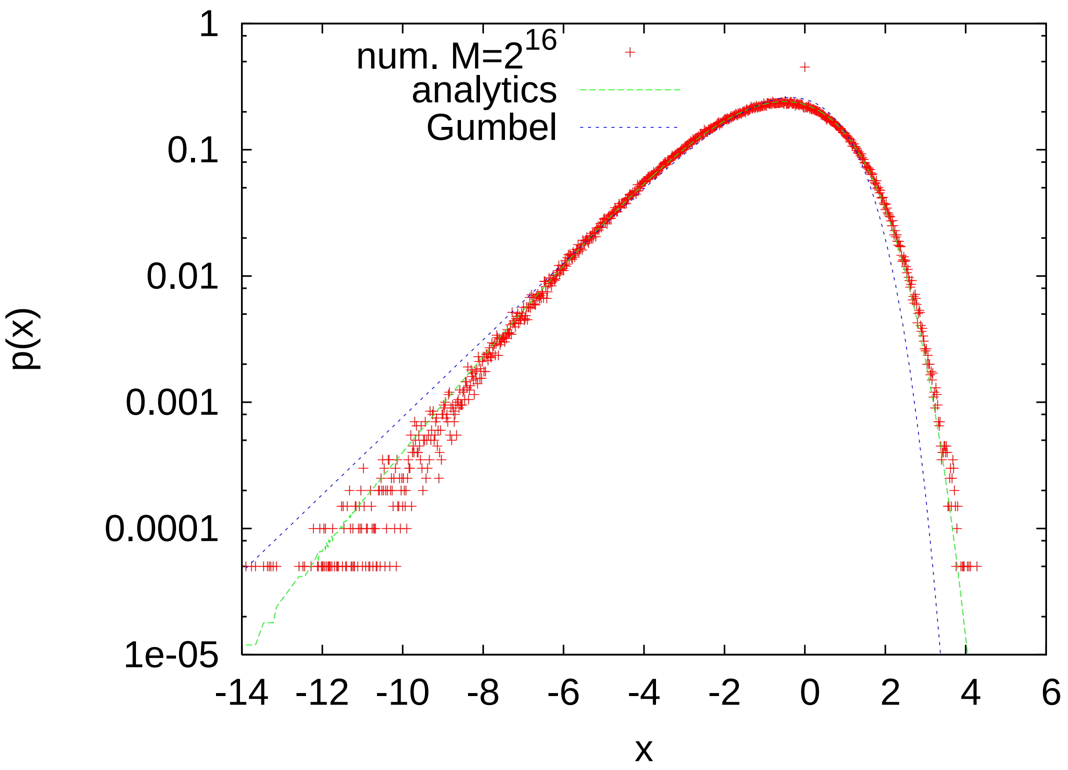
$$= -\frac{d}{df}\left[1+\sum_{n=1}^{\infty}\frac{e^{n\beta_c f}}{n!(n-1)!\Gamma\left(1-n\frac{\beta_c}{\beta}\right)}\left(\beta_c f+\frac{1}{n}-2\psi(n+1)+\frac{\beta_c}{\beta}\psi\left(1-n\frac{\beta_c}{\beta}\right)\right)\right]$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In the zero temperature limit $\beta \to \infty$ the free energy distribution yields the **extreme value probability density.**

The minimum of the random sequence is simply given by $V_{min} = -\lim_{T\to 0} f = const + x$, with known $const$ and the probability density of x related to the frozen profile $g_{\beta_c}(x)$ by

$$p(x) = -g'_{\beta_c}(x) = -\frac{d}{dx}\left[2e^{x/2}K_1(2e^{x/2})\right] \quad (1)$$

This is different from **Gumbel** distribution $p_{Gum}(x) = -\frac{d}{dx}\left[\exp -Be^{Ax}\right]$.



Distribution of extremes: we compare three distributions: (i) the histrogram for ensemble of 10^6 realizations of the periodic **1/f** model sampled at $M = 2^{16}$ equispaced points, (ii) the analytical prediction (1), and (iii) the Gumbel distribution for the mean $\&$ variance given by (1)

Aperiodic 1/f Gaussian noise:

We can repeat all the same procedure for the appropriately regularized **1/f noise** defined on the full line and sampled along an interval of finite length. The calculation is much more involved and requires ability to continue general **Selberg integrals** away from positive integers to the complex plane.

Although the expressions are different from the circle case, they share all the essential features, most importantly **duality relation** $g_\beta(x) = g_{\beta^{-1}}(x)$ for the generation function $g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$. Applying the freezing scenario, we extract the probability density $p(x)$ of **extreme values**. In particular, we find the universal **Carpentier-Le Doussal tail** for

$$p(x \to -\infty) = -g'_{\beta_c}(x \to -\infty) \sim -x e^{x}$$

shared by both distributions. It has its origin in the characteristic tail of the partition function density $P(z \gg 1) \propto 1/z^2$ developed at criticality, with the first moment $< z >$ becoming **infinite**.

Conclusions & Discussions:

- Using the methods of statistical mechanics we were able to extract the explicit expressions for distributions of extrema of the Gaussian 1/f noise, both periodic and aperiodic. The distributions are manifestly non-Gumbel and show universal backward tail.

- We revealed a "duality relation" satisfied by specific generating function of scaled free energies everywhere in the high temperature phase. The same object is expected to show freezing of its shape at the critical temperature. It is tempting to conjecture relation between freezing and self-duality.

- It remains a challenge:

  (i) to verify/justify the freezing scenario (ii) to access extreme value statistics of the two-dimensional object with logarithmic correlations: the Gaussian Free Field in 2D domains.

  Related works in progress:

  i) Statistics of velocities in decaying Burgers turbulence with correlated initial conditions $< v(x)v(x') >\sim |x-x'|^{-2}$.

  ii) Statistics of extrema of characteristic polynomials of random matrices and of Riemann zeta-function.