The critical temperature of dilute Bose gases

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Joint work with Volker Betz (University of Warwick)

Bose-Einstein condensation for ideal gas

Hamiltonian for N bosons in box $\Lambda \subset \mathbb{R}^d$:

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i \qquad \text{in } L^2_{\text{sym}}(\Lambda^N)$$

In Fourier space: $\text{Tr } \mathrm{e}^{-\beta \boldsymbol{H}} = \sum_{(n_k): \sum_k n_k = N} \prod_{k \in \Lambda^*} \mathrm{e}^{-\beta k^2 n_k}$, $\Lambda^* = \frac{2\pi}{L} \mathbb{Z}^d$

Expectation of zero mode:

$$\frac{\langle n_0 \rangle}{V} = \frac{\sum_{(n_k)} n_0 \prod_{k \in \Lambda^*} \mathrm{e}^{-\beta k^2 n_k}}{\text{Tr } \mathrm{e}^{-\beta \boldsymbol{H}}} \quad \longrightarrow \quad \begin{cases} 0 & \text{if } \rho \leqslant \rho_{\text{c}} \\ \rho - \rho_{\text{c}} & \text{if } \rho \geqslant \rho_{\text{c}} \end{cases}$$

with critical density

$$\rho_{\text{c}} = \frac{\zeta(\frac{d}{2})}{(4\pi\beta)^{d/2}}$$

Effects of interactions on critical temperature

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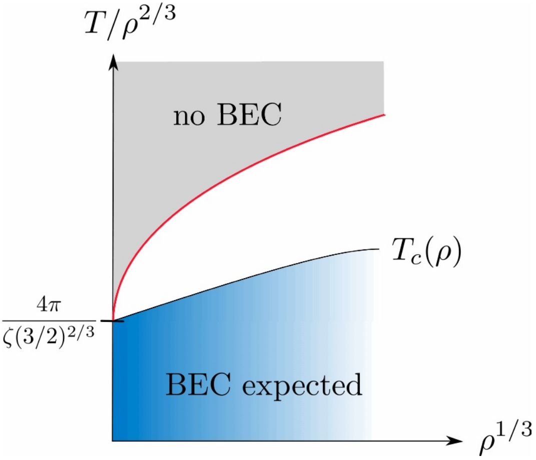
A partial but rigorous result:

Theorem (with Seiringer, 2009)
There is no BEC when

$$\frac{T - T_{\rm c}}{T_{\rm c}} > 5.09 \sqrt{a \rho^{1/3}}$$



Outline of our method

(1) The Feynman-Kac formula yields a model of “spatial random permutations”
(2) Partial mean-field theory for fixed permutation
(3) Exact calculations of cycle weights
(4) Exact calculation of the critical density of the resulting model

Interacting Bose gas: Feynman-Kac representation

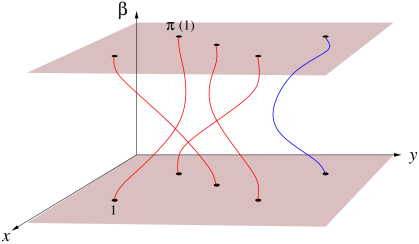
Hamiltonian: $H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i<j \leqslant N} U(x_i - x_j)$ in $L^2_{\mathrm{sym}}(\Lambda^N)$.

Feynman-Kac representation of the partition function (Feynman 1953, Ginibre 1971):

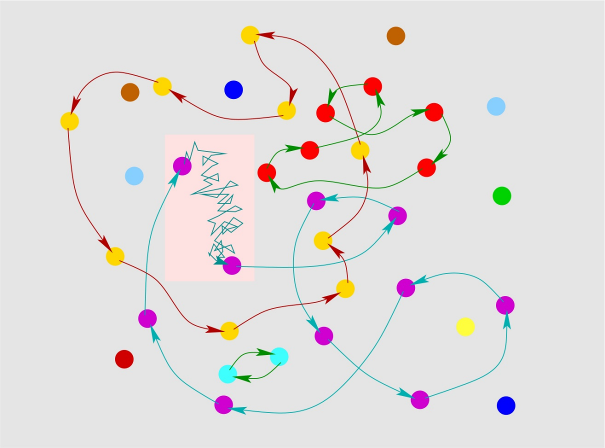
$$\mathrm{Tr}\ \mathrm{e}^{-\beta H} = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \int \mathrm{d}x_1 \dots \mathrm{d}x_N$$

$$\int \mathrm{d}W^{2\beta}_{x_1 x_{\pi(1)}}(\omega_1) \dots \mathrm{d}W^{2\beta}_{x_N x_{\pi(N)}}(\omega_N)$$

$$\exp\Big\{ -\tfrac{1}{2} \sum_{i<j} \int_0^{2\beta} U\big(\omega_i(s) - \omega_j(s)\big) \mathrm{d}s \Big\}$$



Model of spatial random permutations



CONJECTURE 1. The critical temperature for Bose-Einstein condensation is identical to the critical temperature for the occurrence of infinite cycles. (Proved by Sütő ('93, '02) for the ideal gas)

Partial mean-field

Gibbs distribution for points & permutation $(\boldsymbol{x}, \pi)$:

$$\frac{1}{Z}\, \mathrm{e}^{-H_0(\boldsymbol{x},\pi)}\; \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{x},\pi)}$$

with $H_0(\boldsymbol{x}, \pi) = \frac{1}{4\beta} \sum_{i=1}^N |x_i - x_{\pi(i)}|^2$
Let $\mu^{(\pi)}(\mathrm{d}\boldsymbol{x})$ be the point process associated with $\mathrm{e}^{-H_0(\boldsymbol{x},\pi)}$. We replace the Gibbs distribution above with

$$\frac{1}{Z}\, \mathrm{e}^{-H_0(\boldsymbol{x},\pi)}\; \exp\Big\{ -\sum_{i<j} \int V_{ij}(\boldsymbol{y}, \pi)\, \mathrm{d}\mu^{(\pi)}(\boldsymbol{y}) \Big\}$$

CONJECTURE 2. The new model has the same critical temperature as the original model, up to a correction $o(a)$

Model of spatial permutations with cycle weights

Define the weights

$$\alpha_\ell = \sum_{i,j\in\gamma, i<j} \int V_{ij}(\boldsymbol{x},\gamma) \mathrm{d}\mu^{(\gamma)}(\boldsymbol{x}), \qquad \gamma = (2,\dots,\ell,1)$$

$$\alpha_{\ell,\ell'} = \tfrac{1}{2} \sum_{i\in\gamma, j\in\gamma'} \int V_{ij}(\boldsymbol{x},\gamma\cup\gamma') \mathrm{d}\mu^{(\gamma\cup\gamma')}(\boldsymbol{x}),$$

$$\gamma\cup\gamma' = (2,\dots,\ell,1)(\ell+2,\dots,\ell+\ell',\ell+1)$$

We obtain the Hamiltonian

$$H(\boldsymbol{x},\pi) = \frac{1}{4\beta}\sum_i |x_i - x_{\pi(i)}|^2 + \sum_{\ell\geqslant 1} (\alpha_\ell - \alpha_{\ell,\ell}) r_\ell(\pi) + \sum_{\ell,\ell'\geqslant 1} \alpha_{\ell,\ell'} r_\ell(\pi) r_{\ell'}(\pi)$$

where $r_\ell(\pi)$ is the number of ℓ-cycles in the permutation π

Calculations of the cycle weights

Computations give

$$\alpha_{\ell,\ell'} = \frac{4\pi\beta\ell\ell' a}{|\Lambda|}$$

(to first order). Then

$$\sum_{\ell,\ell' \geqslant 1} \alpha_{\ell,\ell'} r_\ell(\pi) r_{\ell'}(\pi) = \frac{4\pi\beta a N^2}{|\Lambda|}$$

is constant in the canonical ensemble. This term can be dismissed!
We also get

$$\alpha_\ell = \frac{\ell a}{(4\pi\beta)^{1/2}} \Big[\sum_{j=1}^{\ell-1} \Big(\frac{\ell}{j(\ell-j)} \Big)^{3/2} - 2\zeta(\tfrac{3}{2}) \Big]$$

(to first order)

Critical density

We can compute the pressure of the model with cycle weights, and its derivative at $\mu = 0$. We get the formula for the critical density:

$$\rho_{\mathrm{c}}^{(a)} = \sum_{\ell \geqslant 1} \frac{\mathrm{e}^{-\alpha_\ell}}{(4\pi\beta\ell)^{3/2}}$$

Is this formula relevant for the occurrence of infinite cycles?

- Sütő proved it for $\alpha_\ell \equiv 0$ ('93, '02)

Here, we have $\alpha_\ell = -\frac{6-3\gamma_{1/2}}{(4\pi\beta)^{1/2}}\, a\,\big(1 + O(\ell^{-1/5})\big)$

- With Betz, we proved it when α_ℓ are small enough ('10)

Correction to the critical temperature

We get

$$\frac{\rho_{\rm c}^{(a)} - \rho_{\rm c}^{(0)}}{\rho_{\rm c}^{(0)}} = -\frac{2a}{(4\pi\beta)^2} \sum_{\ell \geqslant 1} \frac{1}{\ell^{1/2}} \Big[\frac{1}{2} \sum_{j=1}^{\ell-1} \Big(\frac{\ell}{j(\ell-j)} \Big)^{3/2} - \zeta(3/2) \Big]$$

$$= +\frac{2\sqrt{\pi}}{\zeta(3/2)} a\beta^{-1/2}$$

This implies that

$$\frac{T_{\rm c}^{(a)} - T_{\rm c}^{(0)}}{T_{\rm c}^{(0)}} = -\frac{8\pi}{3\zeta(\frac{3}{2})^{4/3}} \, a\rho^{1/3} = -2.33 \, a\rho^{1/3}$$

This contradicts the consensus of the physics literature!!!
(Constant is $+1.3$)

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- Test of the method: the free energy of our simplified model is

$$f(\beta,\rho) = f^{(0)}(\beta,\rho) + 4\pi a[2\rho^2 - (\rho-\rho_{\rm c})_+^2]$$

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- Our method should give the right correction to the critical temperature, but the discrepancy with the physics literature is puzzling