

Classical and Quantum Localization in two and three dimensions

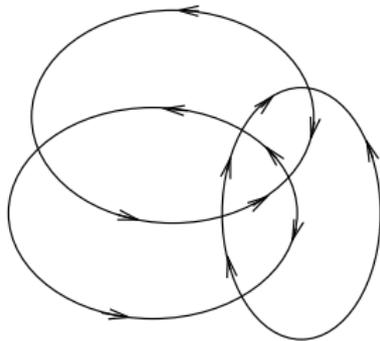
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This talk is about some mathematical results on physical models of particle transport in random media

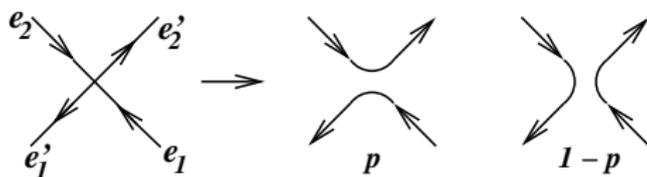
- ▶ propagation in a random medium is often modelled by a *network*:
- ▶ medium represented by some fixed graph \mathcal{G} (initially closed)
 - ▶ for convenience the edges of \mathcal{G} are assumed *oriented*, and each node has exactly 2 incoming and 2 outgoing edges



- ▶ at each tick of the clock, the particle moves through a single node, from an edge e to a neighbouring edge e'

Classical version

- ▶ decompose each node as shown, with probabilities p or $1 - p$:



- ▶ this decomposes \mathcal{G} into a union of cycles (closed loops), so we have a *deterministic* walk in a *random* medium
- ▶ *equivalently*, do not decompose \mathcal{G} , but let the particle, the first time it reaches a given node along (say) edge e_1 , choose to exit via e_1' with probability p , or e_2' with probability $1 - p$
- ▶ the next time it visits the node, it either approaches along e_1 in which case it *must* exit the same way as it did before, or it approaches along e_2 , in which it must exit the other way
- ▶ this is a *history-dependent random* walk

Quantum version

- ▶ a quantum particle described by an N -component wave function moves on the lattice
- ▶ at each tick of the clock it moves from edge e through a node to an edge e'
- ▶ as it moves along edge e its wave function gets multiplied by a $U(N)$ matrix U_e
- ▶ its propagation through the nodes is described by a unitary S -matrix of *amplitudes* $S(e', e)$
- ▶ the amplitude for the particle to propagate from e_1 to e_2 is given by a sum over random walks (Feynman paths Γ), each weighted by a product of U s and S s along the path $\Gamma = (e_1, e'_1, \dots, e_2)$:

$$G(e_2, e_1) = \sum_{\Gamma} U_{e_2}^{1/2} \cdots U(e'_1) S(e'_1, e_1) U_{e_1}^{1/2}$$

- ▶ a *point conductance* measurement corresponds to breaking open a subset $\{e\}$ of edges of \mathcal{G} , $e \rightarrow (e_{\text{in}}, e_{\text{out}})$, choosing a pair of these $(e_{\text{in},1}, e_{\text{out},2})$ and computing

$$\text{Tr } G(e_{\text{in},1}, e_{\text{out},2})^\dagger G(e_{\text{out},2}, e_{\text{in},1})$$

- ▶ the off-diagonal terms between the paths contributing to G^\dagger and G produce *quantum interference* effects
- ▶ in principle the amplitudes $S(e', e)$ and U_e are fixed, but we may view these as drawn from some random ensemble
 - ▶ e.g. we can choose the U_e to be independent random matrices drawn from the invariant measure on $\text{SU}(N)$
- ▶ *self-averaging*: in the thermodynamic limit $|\mathcal{G}| \rightarrow \infty$ many physically interesting quantities are almost surely equal to their mean values

- ▶ for the case $N = 1$ the U_e are pure phases $e^{i\phi_e}$ and $S(e, e')$ are complex numbers, but actually this is harder than the case
- ▶ $N = 2$, where we have the

Theorem. *The mean point conductance $\overline{\text{Tr } G(e_{\text{in}}, e_{\text{out}})^\dagger G(e_{\text{out}}, e_{\text{in}})}$ is given by the probability that there exists an open path from e_{in} to e_{out} in the classical model with $p(e, e') = (1/N)\text{Tr } S(e, e')^\dagger S(e', e)$.*

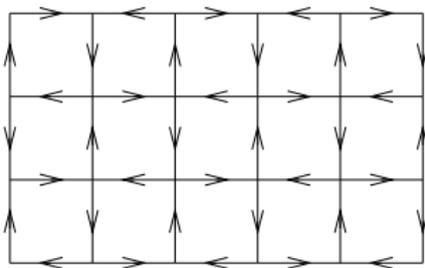
The proof is either exhaustive (Beamond, Chalker, JC 2002) or using a supersymmetric formalism (JC 2005) but in both case depends on a special properties of the invariant measure on $SU(2)$ matrices:

$$\int U^p dU = \begin{cases} 1 & p = 0 \\ -\frac{1}{2} & p = 2 \\ 0 & \text{otherwise} \end{cases}$$

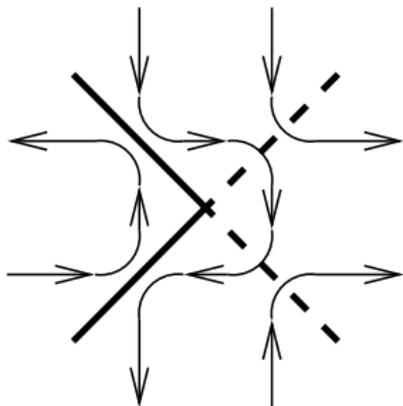
which means that, after averaging, each edge gets traversed either 0 or 2 times.

- ▶ the interesting cases are when \mathcal{G} is a regular lattice in \mathbf{R}^d , in the limit when $|\mathcal{G}| \rightarrow \infty$:
- ▶ if the loops in the decomposition of \mathcal{G} are a.s. of finite length as $|\mathcal{G}| \rightarrow \infty$, the states in the corresponding problem are all *localized*.
- ▶ on the other hand, if there is a finite probability of escape to infinity in the classical problem, the quantum states are *extended*.
- ▶ since classical problems are easier than quantum ones, studying the former may tell us something about the latter
- ▶ the physics literature suggests that
 - ▶ for $d = 2$ states are usually localised except in special cases where an extra symmetry holds, and in $d = 3$ there can be a transition from localized to extended states
 - ▶ in the extended phase, there should be diffusive behaviour, that is the classical paths should scale to Brownian motion
 - ▶ at the transition in $d = 3$, and in the special cases in $d = 2$, one expects power laws in, for example, the distribution of loop lengths

A 2d special case: the L-lattice

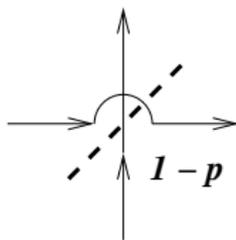
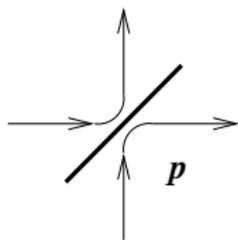
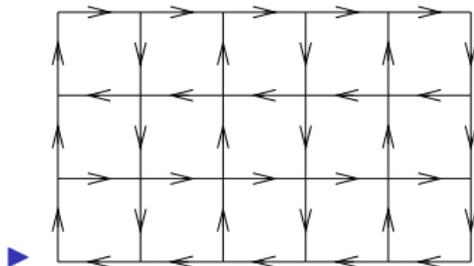


► decompose each node according to



- each decomposition of \mathcal{G} corresponds to a bond percolation configuration on a \mathbf{Z}^2 lattice
- the loops are the external and internal hulls of percolation clusters
- if $p \neq p_c = \frac{1}{2}$ almost all the loops are finite as $|\mathcal{G}| \rightarrow \infty$
- at $p = \frac{1}{2}$ there is a power-law distribution of loop lengths
- corresponding quantum problem is the $SU(2)$ version of the network model for the integer quantum Hall plateau transition (Chalker, Coddington 1988); the connection to bond percolation was first realised by Gruzberg *et al.* 1999.

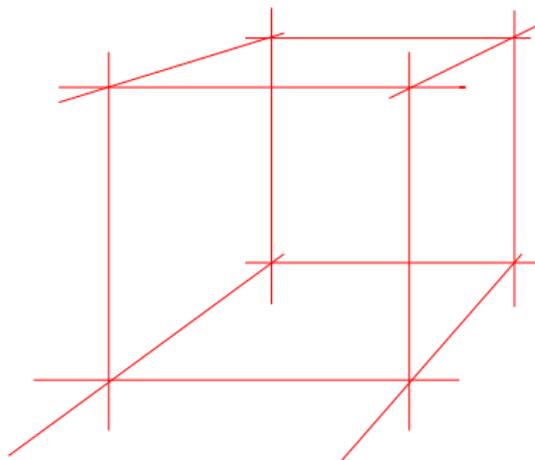
A more generic case: the Manhattan lattice



- ▶ for $p > \frac{1}{2}$ all loops are trapped in finite voids within percolation clusters and are therefore finite
- ▶ in fact, analytic RG calculations and simulations suggest that loops are always finite as long as $p > 0$ (Beamond, Owczarek, JC 2003), consistent with the idea that generically all states are localized in $d = 2$, but there is no proof.

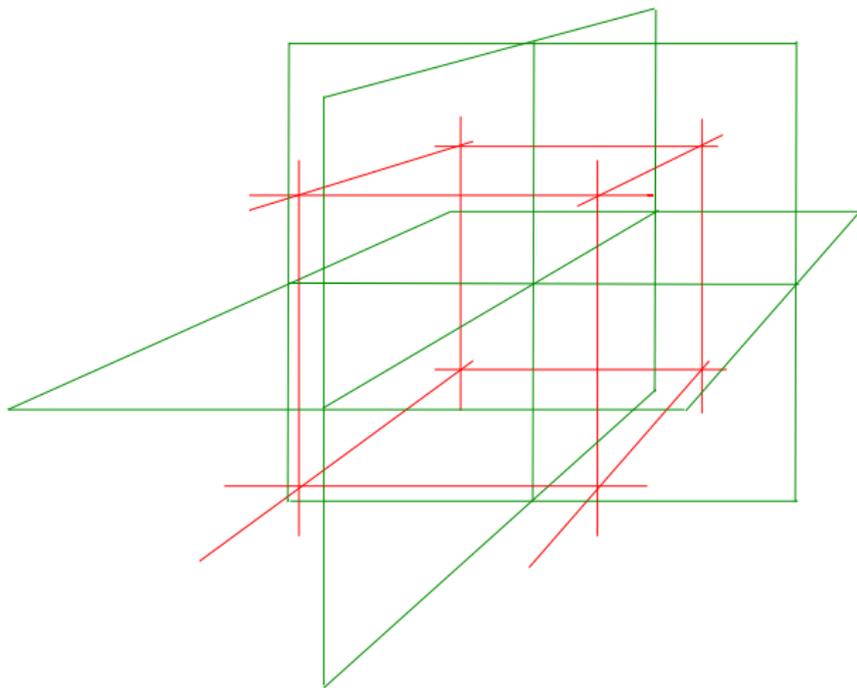
Three dimensions

- ▶ how to build a regular lattice in \mathbf{R}^3 with coordination number 4 and cubic symmetry:

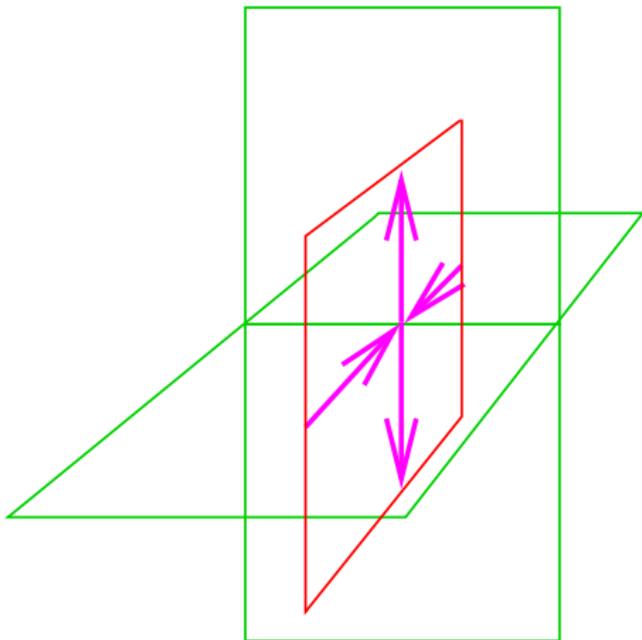


- ▶ start with the faces of \mathbf{Z}^3

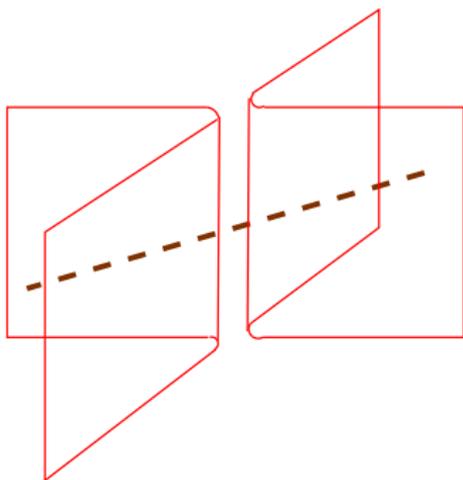
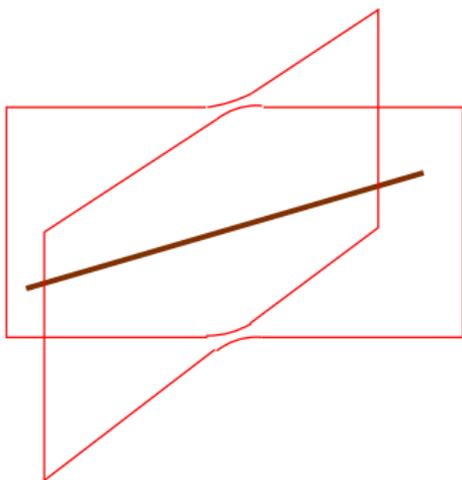
- ▶ add the faces of $(\mathbf{Z} + \frac{1}{2})^3$:



- ▶ \mathcal{G} is the intersection of the two sets of faces



- ▶ bond percolation on one sublattice of \mathbf{Z}^3 (an fcc lattice) decomposes the faces of $(\mathbf{Z} + \frac{1}{2})^3$ into hulls $\{H\}$: closed oriented surfaces which separate the clusters from the dual clusters



- ▶ the loops of \mathcal{G} are the intersection the hulls of bond percolation on two interpenetrating fcc lattices
- ▶ these paths are therefore a.s. finite if $p < p_c^{\text{fcc}} \approx 0.12$
- ▶ this problem has a symmetry under $p \rightarrow 1 - p$; simulations suggest that $p = \frac{1}{2}$ is in the extended phase, but no proof
- ▶ in the Manhattan version of this problem, paths are localised for $p > 1 - p_c^{\text{fcc}}$, and we expect them to be asymptotically like random walks for small p , but no proof yet

Summary

- ▶ random decompositions of graphs into the union of cycles give rise to interesting models of both classical and quantum localisation

Some references:

- ▶ L-lattice: Gruzberg et al., Phys.Rev.Lett.82:4524, 1999
- ▶ main theorem for general graphs: Beamond, JC, Chalker, Phys. Rev. B. 65, 214301, 2002; JC Comm. Math. Phys. 1432-0916, 2005
- ▶ Manhattan lattice: Beamond, JC, Owczarek, J. Phys. A 36, 10251, 2003
- ▶ 3d lattices: Ortuno, Somoza, Chalker, Phys.Rev.Lett. 102, 070603, 2009; JC, in preparation.