Constructing cut and project sets which are close to lattices

Alan Haynes
University of York

Cut and project method

- Suppose that V is a $d-$dimensional subspace of $\mathbb{R}^k$, let

$$\pi : \mathbb{R}^k \to \mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$$

  be the canonical projection, and suppose that $\mathcal{S} \subseteq \mathbb{T}^k$ is a 'nice' set.

- For each $x \in \mathbb{R}^k$ we define the **cut and project set** $Y = Y_{\mathcal{S},x} \subseteq V$ by

$$Y_{\mathcal{S},x} := \{v \in V : \pi(v + x) \in \mathcal{S}\}.$$

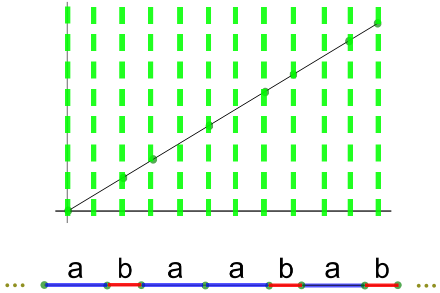
Example (d=1,k=2)

- V the subspace of $\mathbb{R}^2$ generated by the vector

$$\begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix},$$

- $x = 0$ and $\mathcal{S}$ the line segment in the plane from $(0, 2-\sqrt{5})$ to $(0, (3-\sqrt{5})/2)$, $\ldots$

Fibonacci tiling



$$a \mapsto ab$$
$$b \mapsto a$$

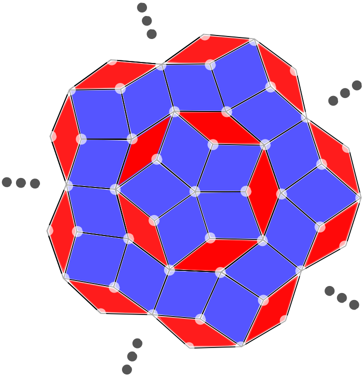
Example (d=2,k=5)

- V the subspace of $\mathbb{R}^5$ generated by the columns of the matrix

$$\begin{pmatrix} 1 & 0 \\ \cos(2\pi/5) & \sin(2\pi/5) \\ \cos(4\pi/5) & \sin(4\pi/5) \\ \cos(6\pi/5) & \sin(6\pi/5) \\ \cos(8\pi/5) & \sin(8\pi/5) \end{pmatrix},$$

- $\mathcal{S}$ suitably chosen...

Penrose tiling



Working assumptions

- We will assume that:
  - (i) The subspace V is **totally irrational** (i.e. $\pi(V)$ is dense in $\mathbb{T}^k$), and
  - (ii) The set $\mathcal{S}$ is the injective image under π of a convex, bounded subset of a $(k-d)$-dimensional plane in $\mathbb{R}^k$ which is everywhere transverse V (what we will call a **section**).
- These conditions ensure that $Y_{\mathcal{S},x}$ is **uniformly discrete** and **relatively compact** (i.e. a **Delone set**).

Definitions of BL and BD

- We will say that two sets in $\mathbb{R}^d$ are **bi-Lipschitz (BL)** to one another if there is a bi-Lipschitz bijection between them.
- We will say that two sets in $\mathbb{R}^d$ are **bounded distance (BD)** to one another if there is a bijection between them which moves each point by at most some constant amount.
- Remark 1: For Delone sets, BD implies BL.
- Remark 2: Any two lattices in $\mathbb{R}^d$ of the same covolume are BD to one another.

Motivating questions

- Are all Delone sets BL (or even BD) to a lattice?
  - [Burago, Kleiner; McMullen, 1998] No. There are Delone sets which are not BL to a lattice.
- Are all cut and project sets BL to a lattice?
- Which cut and project sets are BD to a lattice?

Previous results

- [Solomon, 2010] Centers of tiles in Penrose tilings are BD to a lattice.
- [Solomon; H, Kelly, Weiss, 2010] There are cut and project sets which are not BD to any lattice.
- [H, Kelly, Weiss, 2010] For any d, k, for almost every V, if $\dim \partial\mathcal{S} < k - d$ then $Y_{\mathcal{S},x}$ is BL to a lattice.
- [H, Kelly, Weiss, 2010] For $d > 1$, for almost every V, if $\mathcal{S}$ is an aligned box then $Y_{\mathcal{S},x}$ is BD to a lattice.

New result for co-dimension 1

Theorem (H, 2013)

For any $k > 1$ and for any co-dimension 1 subspace V of $\mathbb{R}^k$, there is an infinite collection of sections with the property that for any section $\mathcal{J}$ from the collection, the cut-and-project set $Y_{\mathcal{J},x}$ is distance at most $K|\mathcal{J}|^{-1}$ from a lattice. The constant K may depend on V, but does not depend on x or $\mathcal{J}$.

New result for arbitrary co-dimension

Theorem (H, Koivusalo, 2014)

For any d, k, and V as above, there is an infinite, non-trivial collection of sections with the property that, for any section $\mathcal{S}$ from the collection, the set $Y_{\mathcal{S},x}$ is BD to a lattice.

Bounded remainder sets for toral rotations

- Suppose that $\alpha \in \mathbb{T}^s$ is totally irrational. We say that a measurable set $A \subseteq \mathbb{T}^s$ is a **bounded remainder set (BRS)** for α if

$$\sup_{x \in \mathbb{T}^s} \sup_{N \in \mathbb{N}} \left| \sum_{n=1}^{N} \chi_A(x + n\alpha) - N|A| \right| < \infty.$$

- [Schmidt, 1974] For any α, the set of all volumes of BRS's for α is countably infinite.

Known results

- [Hecke, 1922; Ostrowski, 1930; Kesten, 1966] For $s = 1$ an interval $\mathcal{I}$ is a BRS for α if and only if $|\mathcal{I}| \in \alpha\mathbb{Z} + \mathbb{Z}$.
- [Szüsz, 1954] For $s = 2$, constructed infinite collections of parallelograms which are BRS's.
- [Liardet, 1987] For any s, the only examples of BRS's which are aligned boxes are the trivial examples arising from the $s = 1$ problem.
- [Zhuravlev, 2012] For $s \geq 2$, constructed some non-trivial examples of BRS's using exchanged toric developments.

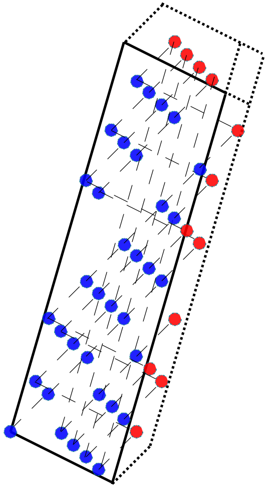
Our result

Theorem (H, Koivusalo, 2014)

We provide a simple algorithm generalizing the continued fraction algorithm, which for any s and any totally irrational $\alpha \in \mathbb{R}^s$ constructs infinitely many non-trivial examples of BRS's.

- Our proof relies on a result of Rauzy (1972), which provides a sufficient condition for a set to be a BRS.

Example of a special region



Lattice above a special region