Random Noble Means Substitutions

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(joint work with Markus Moll)

Motivation

General theme

- What is *order* ? What is *disorder* ?
- How can we *detect* order / disorder ?

  Fourier methods / entropy
- How can we *quantify* order ?

  geometric tools, spectral theory

Specific questions

- How random can order be ?

  Rudin–Shapiro sequence
- How ordered can randomness be ?

  random inflation rules

Symbolic side

Noble means (NM) substitutions $(m \in \mathbb{N},\ 0 \leqslant i \leqslant m)$

$$\zeta_{m,i} : \begin{array}{lcl} \mathtt{a} & \mapsto & \mathtt{a}^i \mathtt{b} \mathtt{a}^{m-i} \\ \mathtt{b} & \mapsto & \mathtt{a} \end{array} \qquad M_m = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}$$

Properties

- $\lambda_m^{\mathrm{PF}} = \frac{1}{2}\bigl(m + \sqrt{m^2 + 4}\,\bigr) = [m; m, m, \ldots]$ is PV
- Hull (LI class) depends only on m, denoted $\mathbb{X}_m$
- Linear complexity $\Rightarrow$ vanishing top. entropy
- $(\mathbb{X}_m, \mathbb{Z}, \mu)$ is strictly ergodic dynamical system
- Pure point dynamical and diffraction spectrum

Word frequencies

PF theory ($m = 1$, $i = 1$, length $\ell = 2$, $\texttt{bb}$ is illegal)

$$\left.\begin{array}{lllll} \texttt{aa} & \mapsto & \texttt{abab} & \curvearrowright & (\texttt{ab})(\texttt{ba}) \\ \texttt{ab} & \mapsto & \texttt{aba} & \curvearrowright & (\texttt{ab})(\texttt{ba}) \\ \texttt{ba} & \mapsto & \texttt{aab} & \curvearrowright & (\texttt{aa}) \end{array}\right\} \quad \Rightarrow \quad M_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Results

- $\lambda_1^{\mathrm{PF}} = \tau = \frac{1}{2}(1+\sqrt{5}\,)$, eigenvalues τ, $\tau' = 1 - \tau$ and 0
- Relative word frequencies from right PF eigenvector

$$\boxed{\nu_{\texttt{aa}} = \tau^{-3}\,,\ \nu_{\texttt{ab}} = \nu_{\texttt{ba}} = \tau^{-2}}$$

- Unique measure μ from cylinder sets via legal words
- Simple method, recursive in ℓ (Queffélec, 1997)

Geometric side

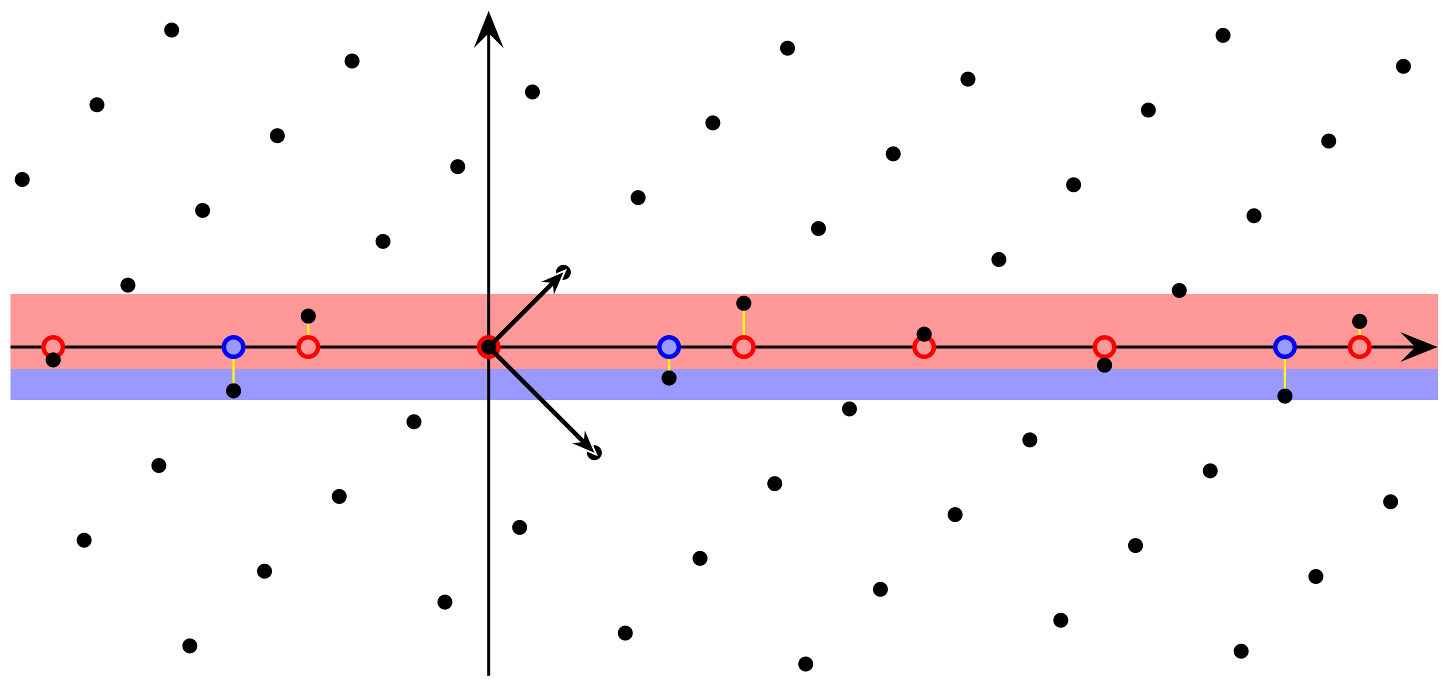
Geometric inflation $(m = 2,\ i = 1,\ \lambda_2^{\mathrm{PF}} = \lambda_2 = 1 + \sqrt{2})$

Properties

- Intervals (prototiles) via left PF eigenvector
- Left endpoints form Delone sets, and Meyer sets
- Continuous hull depends only on m, denoted $\mathbb{Y}_m$
- $(\mathbb{Y}_m, \mathbb{R}, \mu)$ is strictly ergodic dynamical system
- Pure point dynamical and diffraction spectrum

Projection

Model set description (with $L = \mathbb{Z}[\sqrt{2}\,]$, seed $\mathtt{a}|\mathtt{a}$)



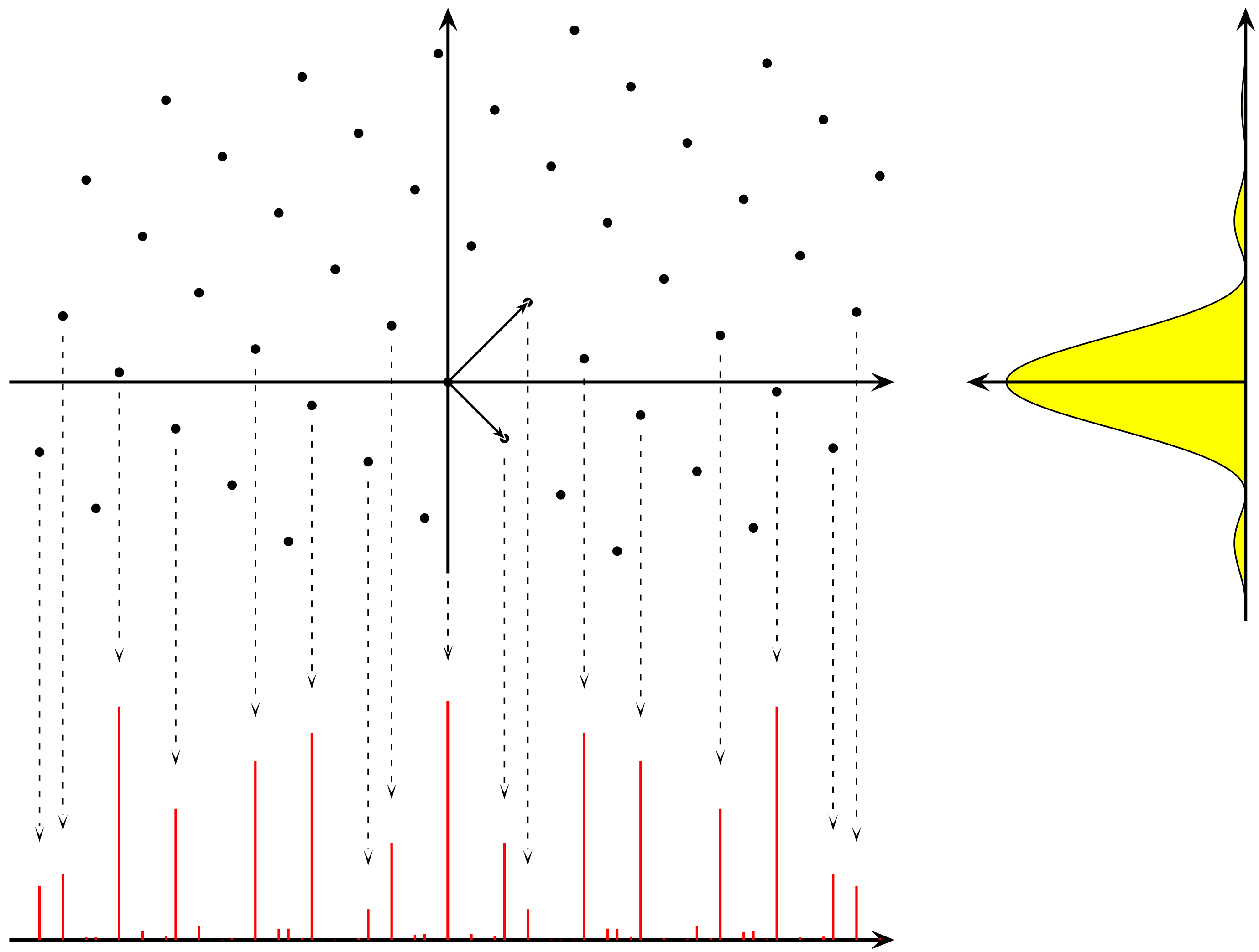
Lattice $\mathcal{L} = \{(x, x') \mid x \in L\}$

Model set $\mathit{\Lambda} = \{x \in L \mid x' \in W\}$

Star map $' : \sqrt{2} \mapsto -\sqrt{2}$

Window $W = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\,\right]$

Silver mean diffraction



General CPS

$$
\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\;\pi\;} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\;\pi_{\mathrm{int}}\;} & \mathbb{R} \\
\text{dense} \;\cup & & \cup & & \cup\; \text{dense} \\
\mathbb{Z}[\lambda_m] & \xleftarrow{\;1-1\;} & \mathcal{L} & \xrightarrow{\;1-1\;} & \mathbb{Z}[\lambda_m] \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}
$$

Properties

- Lattice $\mathcal{L} = \{(x, x^\star) \mid x \in L]\}$
- Star map * is algebraic conjugation in $\mathbb{Q}(\lambda_m)$
- Model set $\varLambda_m = \{x \in L \mid x^\star \in W\}$
- Window $W = [\lambda_m^\star, 1)$ for $i = 0$, seed a|a

Random substitutions

Fix $m \in \mathbb{N}$, choose probability vector $\boldsymbol{p}_m = (p_0, \ldots, p_m) \gg 0$

Define RNM substitution

$$\zeta_m : \begin{cases} \mathtt{a} & \mapsto \begin{cases} \zeta_{m,0}(\mathtt{a}), & \text{with probability } p_0, \\ \quad\vdots & \qquad\qquad\vdots \\ \zeta_{m,m}(\mathtt{a}), & \text{with probability } p_m, \end{cases} \\ \mathtt{b} & \mapsto \quad \mathtt{a}, \end{cases}$$

Properties

- Substitution matrix $M_m = \left(\begin{smallmatrix} m & 1 \\ 1 & 0 \end{smallmatrix}\right)$, PF eigenvalue λ_m
- Hull (via infinite legal words) is much bigger
- Geometric interpretation via intervals, as before

Complexity

Topological entropy

$$h_{\text{top}} = \lim_{n\to\infty} \frac{1}{n} \log\bigl(\text{card}\{\text{legal subwords of length } n\}\bigr)$$

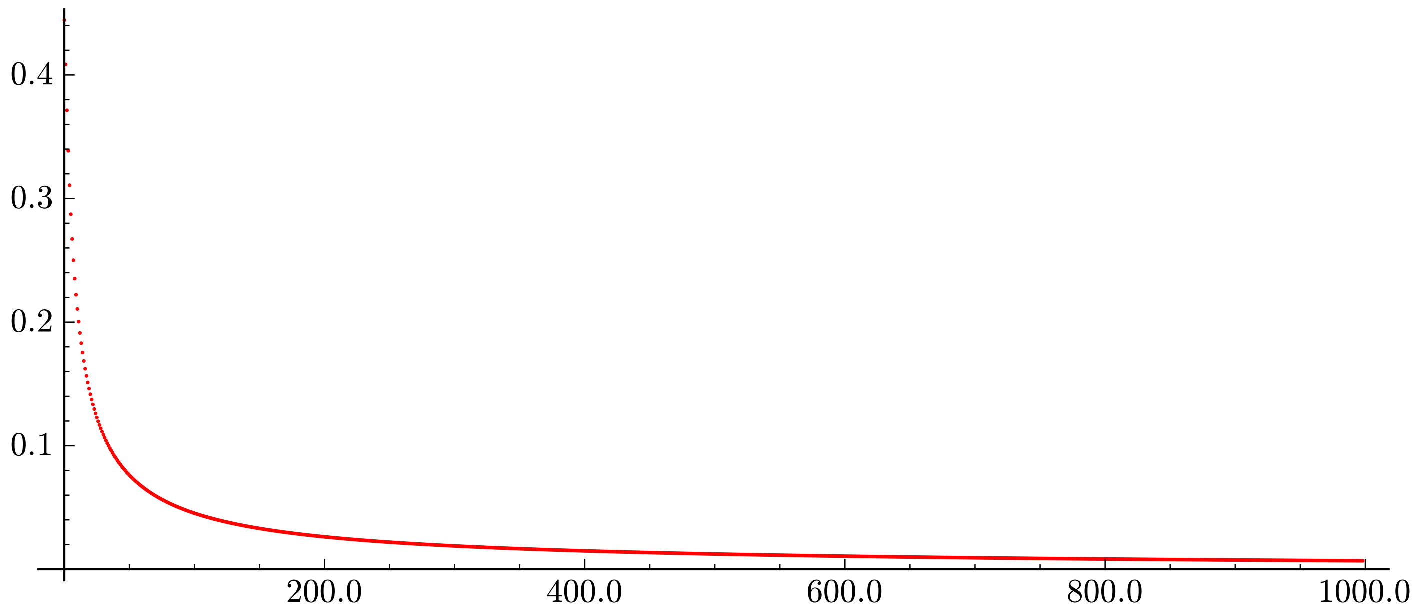
Theorem (Nilsson, 2012; Moll, 2013)

The topological entropy equals the exponential growth rate of the set of exact random inflation words, which is

$$\boxed{h^{(m)}_{\text{top}} = \frac{\lambda_m - 1}{1-\lambda'_m} \sum_{i=2}^{\infty} \frac{\log\bigl(m(i-1)+1\bigr)}{\lambda_m^i} > 0}$$

In particular: $h^{(1)}_{\text{top}} = \sum_{i=2}^{\infty} \frac{\log(i)}{\tau^{i+2}} \approx 0.444399 > 0$ (G & L, 1987)

Entropy



Properties

- $h_{\mathrm{top}}^{(m)}$ is strictly decreasing in m
- One has $\boxed{\lim_{m\to\infty} h_{\mathrm{top}}^{(m)} = 0}$

Ergodicity

Measure construction

- Probabilistic counterpart of PF theory
- Define invariant measure μ via cylinder sets
- Word frequencies exist almost surely (via SLLN)

Theorem (B & M, 2013) The measure μ is ergodic.

Example (m fixed, $\ell = 2$)

$$M_m^{(2)} = \begin{pmatrix} m-1+p_0 p_m & m-1+p_0 & 1-p_0 & 1 \\ 1-p_0 p_m & 1-p_0 & p_0 & 0 \\ 1-p_0 p_m & 1 & 0 & 0 \\ p_0 p_m & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\{\lambda_m, \lambda'_m, p_0 p_m, -p_0\}$

Projection setting

- Geometric counterpart as in deterministic case
- Use general CPS for fixed $m \in \mathbb{N}$
- Observation: Lift ($\star$-image) remains bounded
- $\Rightarrow$ Every inflation sequence is a subset of a model set
- Extension to entire continuous hull via orbit closures

Theorem (G & L, 1987; B & M, 2012)
Every member of the continuous RNM hull is a Meyer set.

Consequence The pure point part of the diffraction is non-trivial, with relatively dense support (Strungaru, 2013)
$\Rightarrow$ System is ergodic, but not weakly mixing

Random diffraction

- Stochastic Dirac comb $\omega = \delta_{\Lambda}$
- Autocorrelation $\boxed{\gamma = \omega \circledast \widetilde{\omega}}$
- Ergodicity $\Rightarrow$ γ is almost deterministic

Diffraction (a.s.) $\boxed{\widehat{\gamma} = \big(\widehat{\gamma}\big)_{\mathsf{pp}} + \big(\widehat{\gamma}\big)_{\mathsf{sc}} + \big(\widehat{\gamma}\big)_{\mathsf{ac}}}$

where

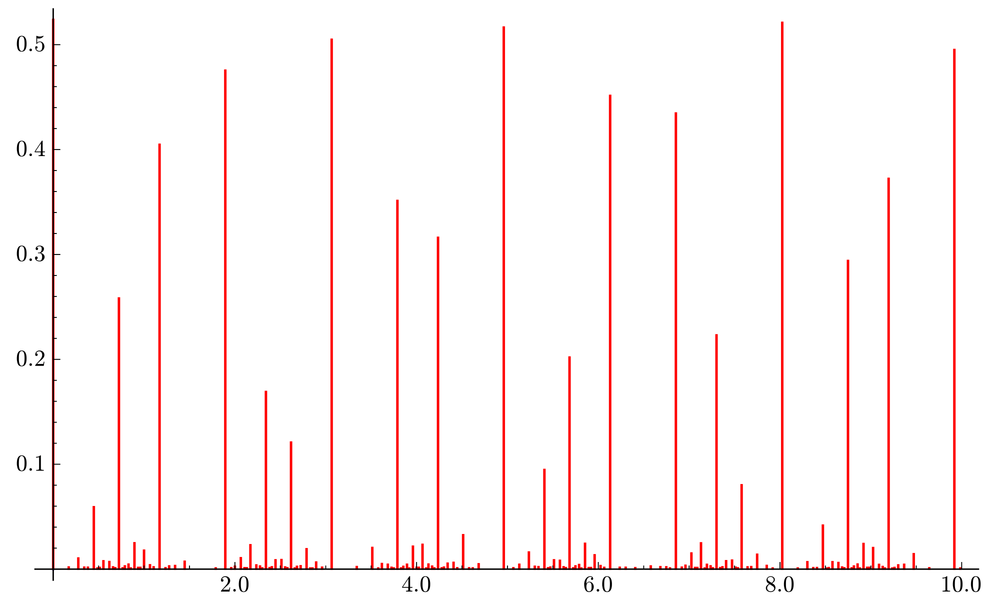
$\big(\widehat{\gamma}\big)_{\mathsf{pp}} = \sum_{k\in\mathcal{F}} |A(k)|^2 \, \delta_k$, with $A(k)$ as the limit of an explicit ensemble recursion

$\big(\widehat{\gamma}\big)_{\mathsf{ac}} = \varPhi \, \lambda_{\mathrm{Leb}}$, with $\varPhi$ a bounded continuous function on $\mathbb{R}$ in series form

$\big(\widehat{\gamma}\big)_{\mathsf{sc}} = 0$ (proof not yet complete)

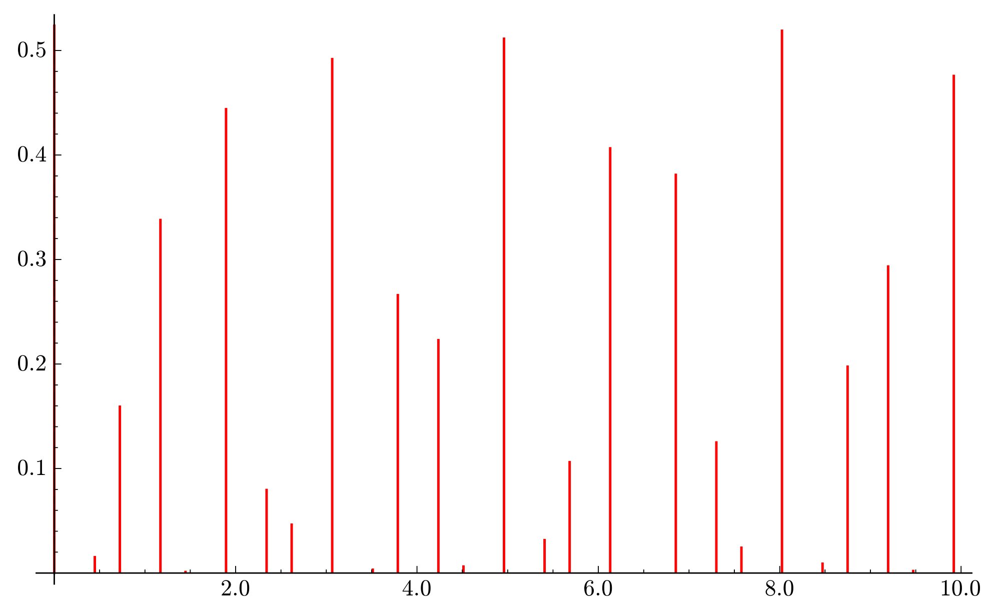
Diffraction, ctd.

perfect Fibonacci

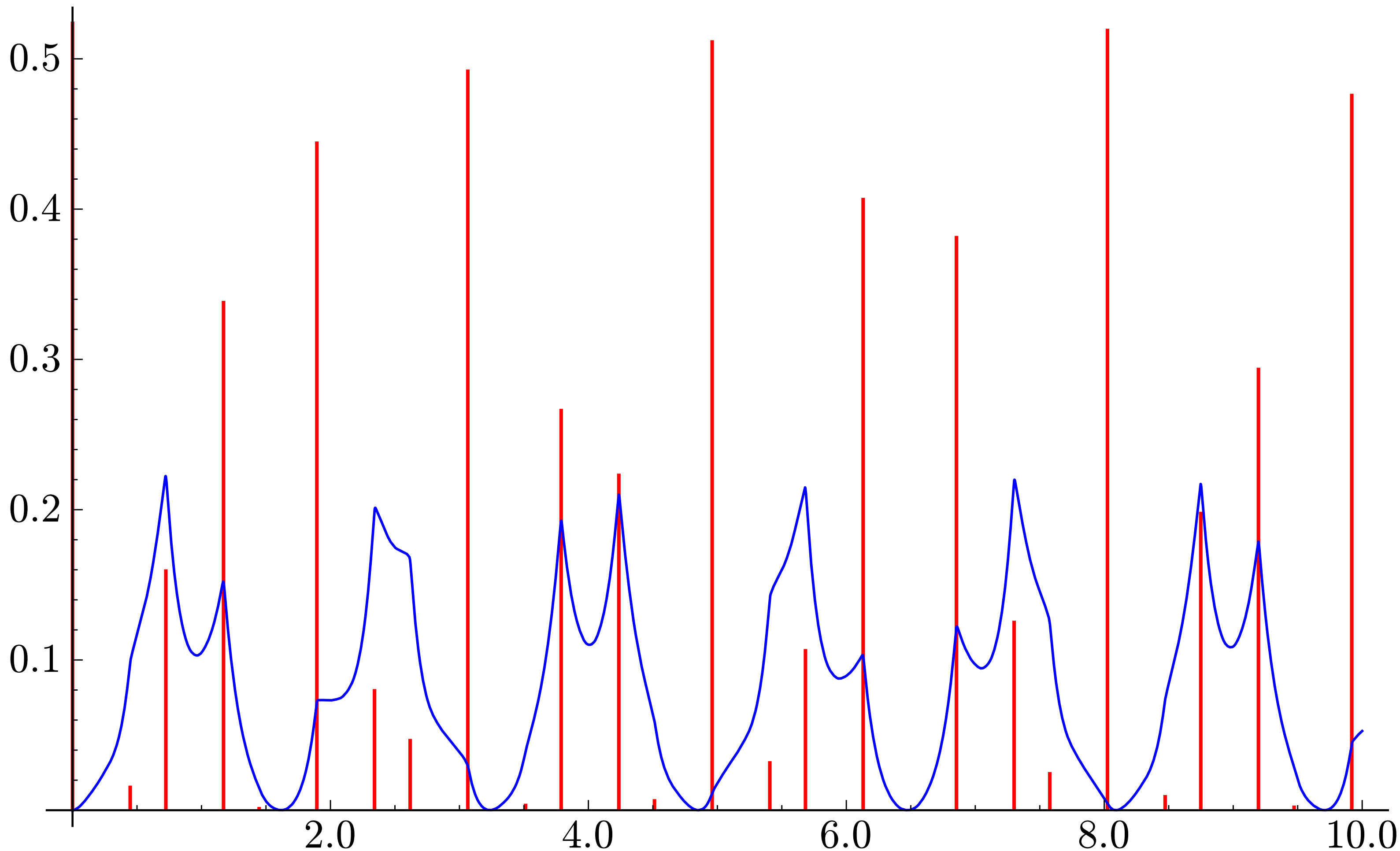


random Fibonacci

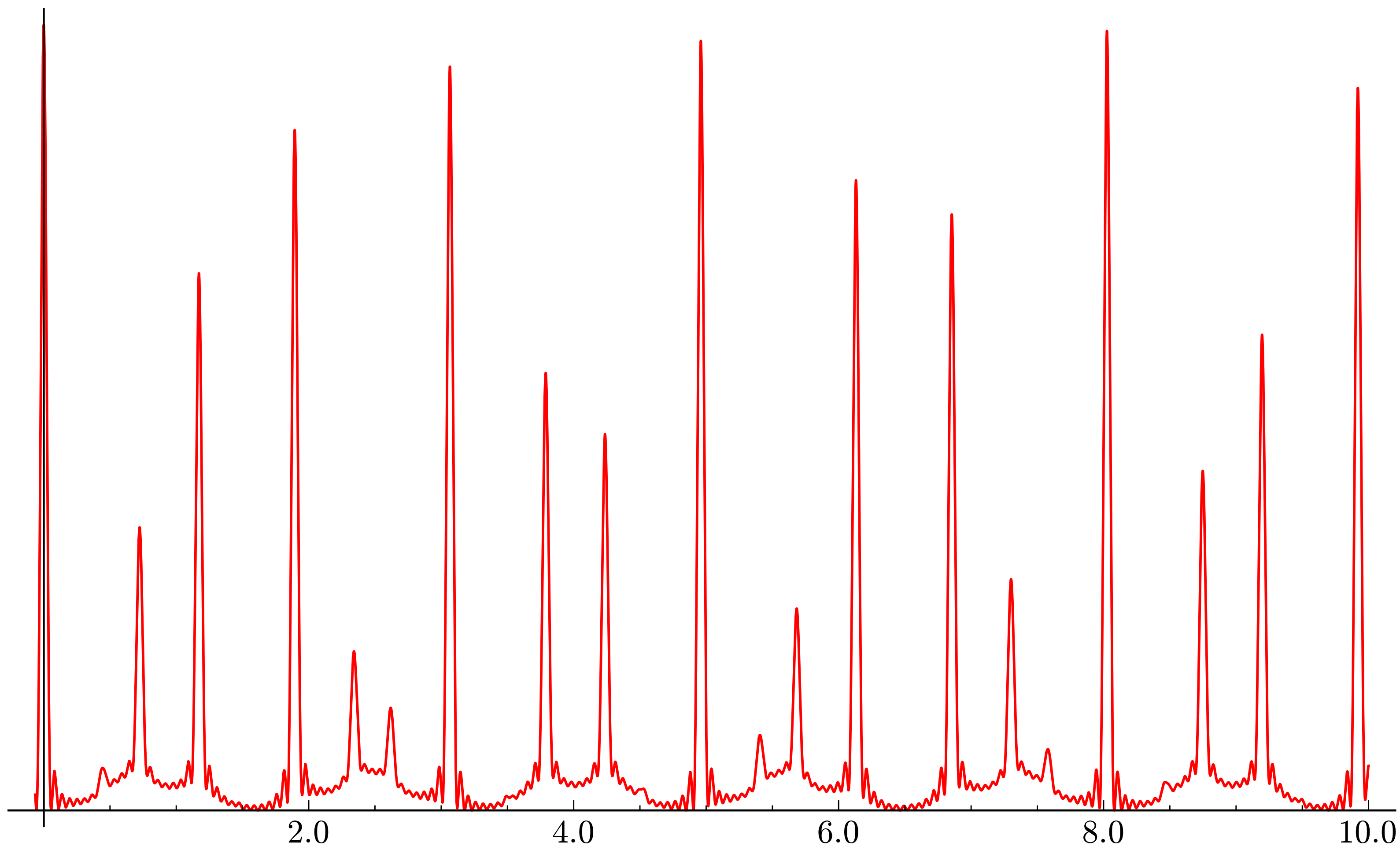
pure point part



Diffraction, ctd.



Diffraction, ctd.



Summary

- Probabilistic extension of substitutions
- Interesting Meyer sets with entropy
- Systems with mixed (diffraction) spectra
- Ergodicity, but absence of weak mixing
- Plethora of further extensions, unexplored
- Counterparts with (planar) tilings exist
- Coexistence of *order* and *randomness*

What is order ? What is disorder ?

To be ctd.

References

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