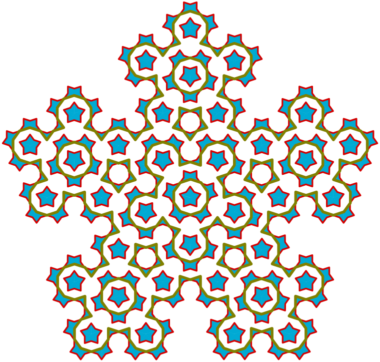
Poincaré duality for pattern-equivariant (co)homology

Jamie Walton

University of Leicester



Outline

1. Pattern-Equivariant Homology
   - Three Spaces of Tilings
   - Cellular PE Homology
   - Poincaré Duality

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   - Example:Penrose Kite and Dart

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   - Example:Penrose Kite and Dart
3. Spectral Sequence for Rotational Tiling Spaces

Outline

Three Spaces of Tilings

Definition (The Translational Hull Ω_1)

Let T be a tiling of $\mathbb{R}^d$ with FLC wrt translations. Define $\Omega_1 := \overline{T + \mathbb{R}^d}$.

Definition (The Euclidean Hull Ω_{rot})

Let T be a tiling of $\mathbb{R}^d$ with FLC wrt rigid motions. Define $\Omega_{\mathrm{rot}} := \overline{E^+(d)(T)}$.

The completions are taken wrt a sensible metric on these sets of tilings. Points of these completions may be identified with tilings which are *locally isomorphic* to T, that is, tilings which have identical finite patches of tiles. The group $SO(d)$ acts on the space Ω_{rot} in the obvious way, by rotations.

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Definition (The Hull of Tilings Modulo Rotations Ω_0)

Let T be a tiling of $\mathbb{R}^d$ with FLC wrt rigid motions. Define $\Omega_0 := \Omega_{\mathrm{rot}}/SO(d)$.

(Cellular) Pattern-Equivariant Homology

Let T be a cellular tiling of $\mathbb{R}^d$, with set of (oriented) k-cells T^k. We define:

$$C_k^{\mathcal{T}_i}(\mathbb{R}^d; G) := \{\sigma : T^k \to G \mid \sigma \text{ is } \mathcal{T}_i\text{-equivariant}\}$$

where such a σ is said to be $\mathcal{T}_i$-*equivariant* ($i = 1, 0$, resp.) if there exists some radius R such that, whenever there is a translation (rigid motion, resp.) taking the patch of tiles within radius R at a k-cell c to the analogous such patch at d, then $\sigma(c) = \sigma(d)$ (taking into account orientations).

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The usual cellular boundary map makes $C_\bullet^{\mathcal{T}_i}(\mathbb{R}^d; G)$ a chain complex with homology groups the *$\mathcal{T}_i$-equivariant homology groups* or *pattern-equivariant homology groups*:

$$H_\bullet^{\mathcal{T}_i}(\mathbb{R}^d; G).$$

Remark

- The chain groups have the analogous description to the cochain groups for PE cohomology (see Kellendonk (2003), Sadun (2007)); however, we consider here the boundary and not the coboundary maps.
- One may also define a singular version. These groups can be defined for "patterns" more general than just tilings (see Pattern-Equivariant Homology paper on the arXiv).
  For the $\mathcal{T}_0$E homology groups, one only gets agreement, in general, between the singular and cellular groups if the cells have trivial local symmetry in the pattern e.g., for $d = 2$ points of local rotational symmetry should be contained in the 0-skeleton.
- One may define $\mathcal{T}_{\mathrm{rot}}$E homology on the Euclidean group $E^+(d)$ in the expected way.

Poincaré Duality

Theorem (W)

Let T be an FLC (wrt translations) tiling of $\mathbb{R}^d$. There is an isomorphism:

$$\check{H}^k(\Omega_1; G) \cong H^{\mathcal{T}_1}_{d-k}(\mathbb{R}^d; G).$$

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- $\check{H}^k(\Omega_0; \mathbb{R}) \cong H^{\mathcal{T}_0}_{d-k}(\mathbb{R}^d; \mathbb{R})$.
- $\check{H}^k(\Omega_{rot}; G) \cong H^{\mathcal{T}_{rot}}_{d-k}(\mathbb{R}^d; G)$.

The above duality does not hold for $\mathcal{T}_0$ over general (non-divisible) coefficients. One can, however, modify the PE homology groups (by changing the "allowed coefficients" at cells with rotational symmetry in the pattern) so as to regain Poincaré duality.
For a tiling of $\mathbb{R}^2$, have SES:

$$0 \to \check{H}^2(\Omega_0) \to H_0^{\mathcal{T}_0}(\mathbb{R}^2) \to \bigoplus_{T_i} \mathbb{Z}_{n_i} \to 0$$

where the direct sum of torsion groups is taken over all tilings T_i in the hull with rotational symmetry of order n_i.

Hierarchical Tilings

Given a substitution rule ω, we say that T is an ω-*hierarchical tiling* if every finite patch of the tiling is a sub-patch of some iteratively substituted tile.

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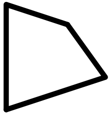
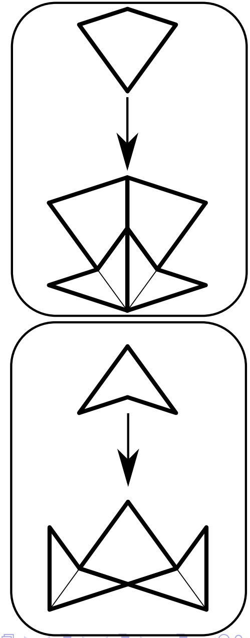
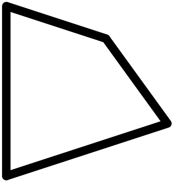
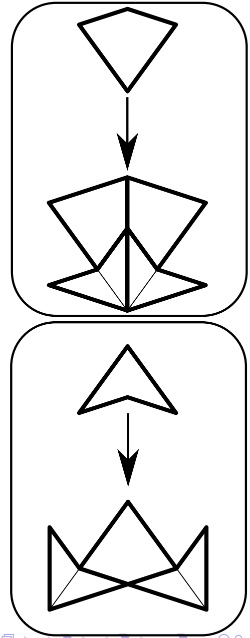
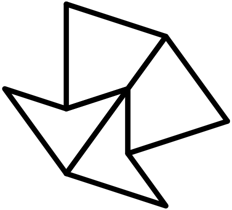
The substitution acts on tilings, and one can show that the map ω is surjective on the collection of all ω-substitution tilings. That is, for every substitution tiling T, there is a (rescaled) version which subdivides to that tiling. We call this tiling a *super-tiling* of T (see Anderson and Putnam (1995)).

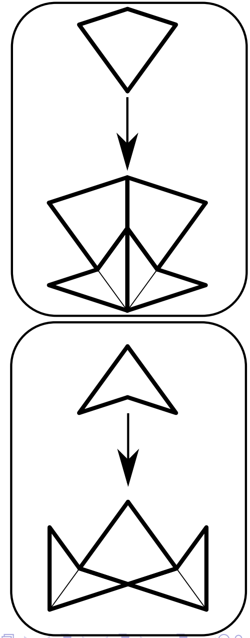


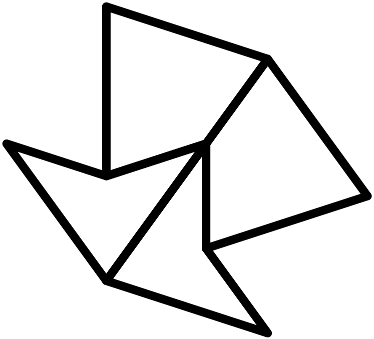


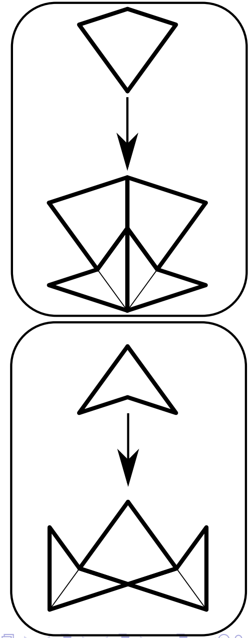


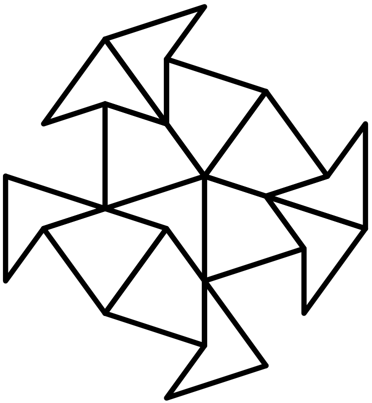


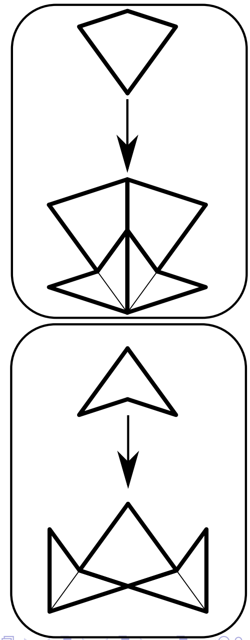


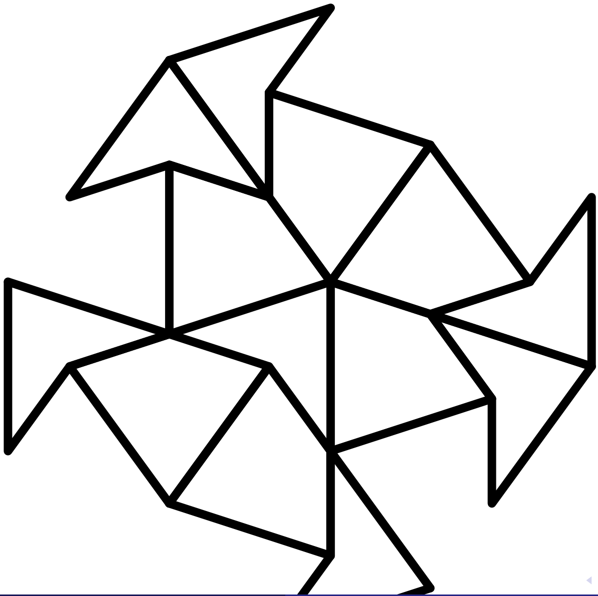


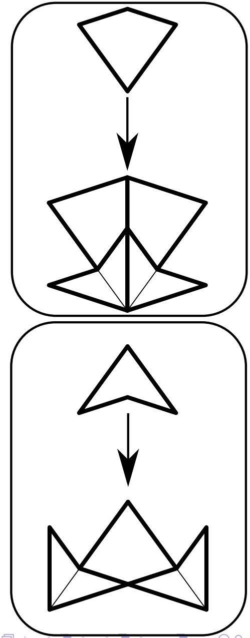


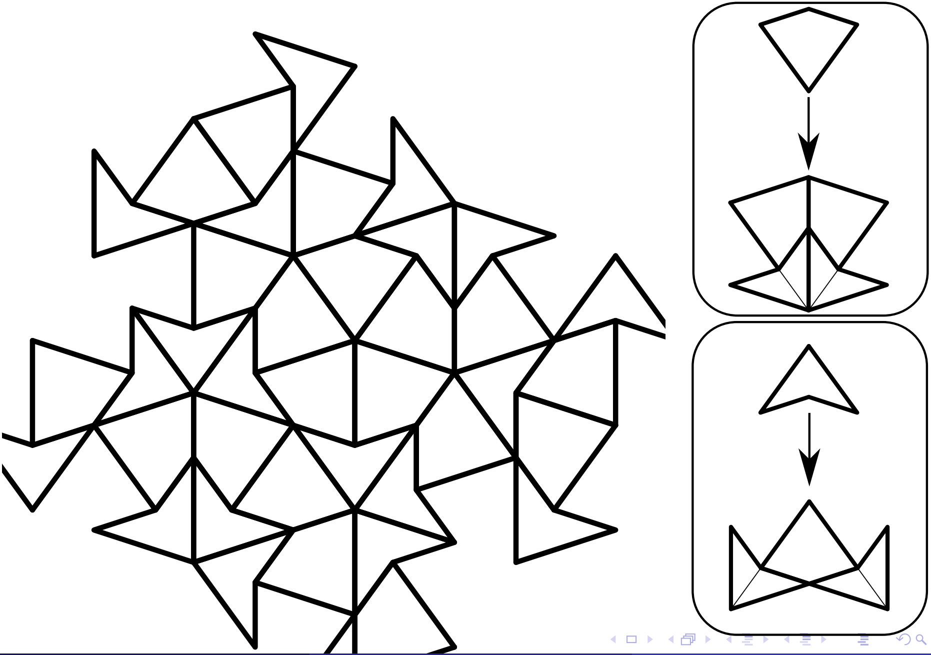


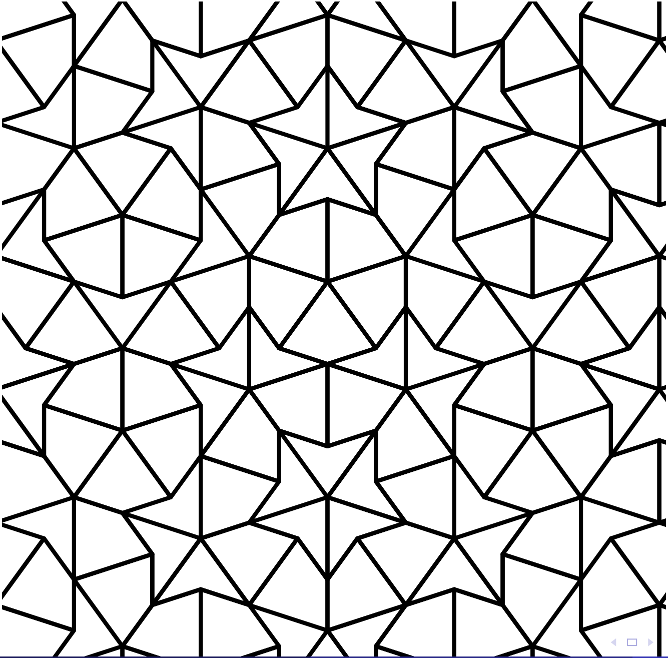


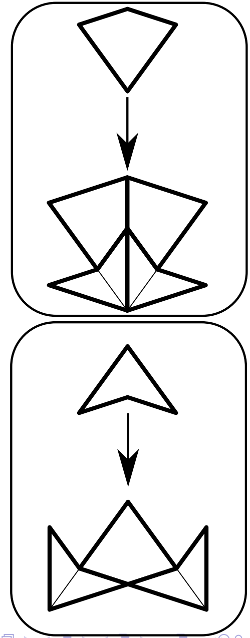


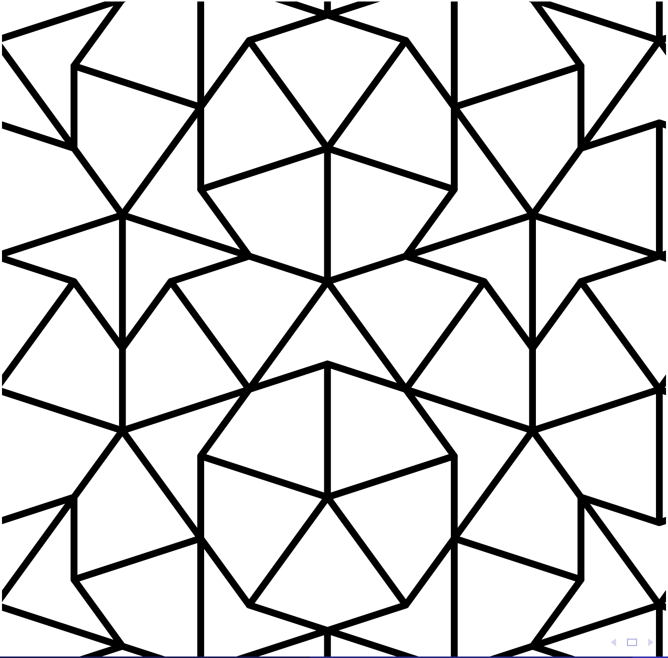


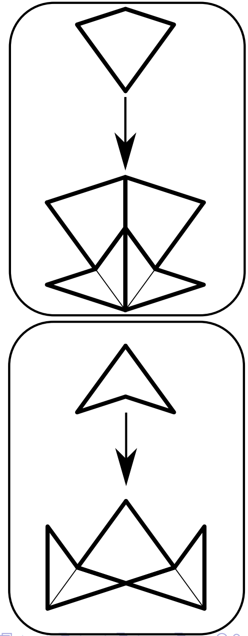


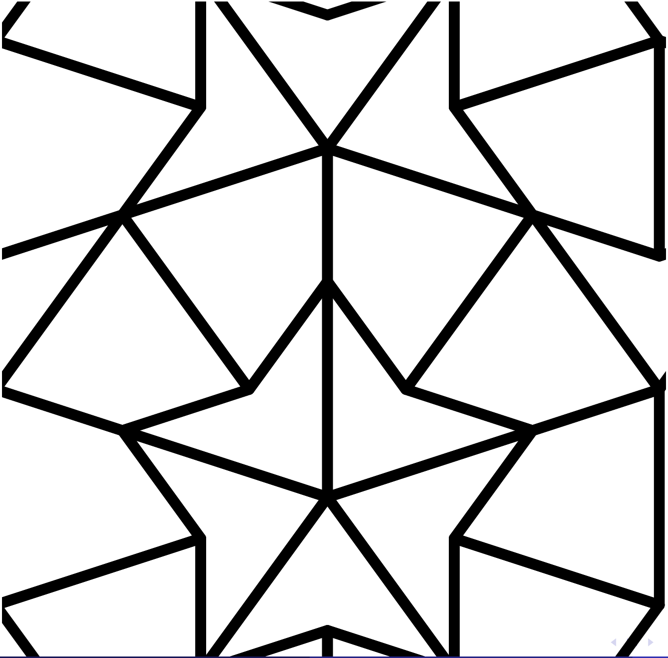


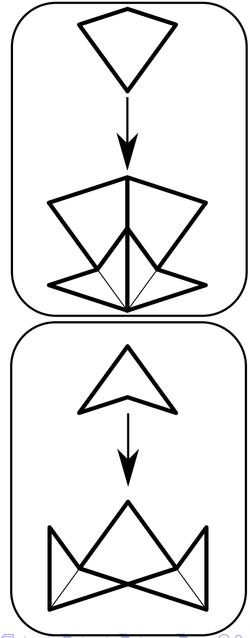












PE Homology of Hierarchical Tilings

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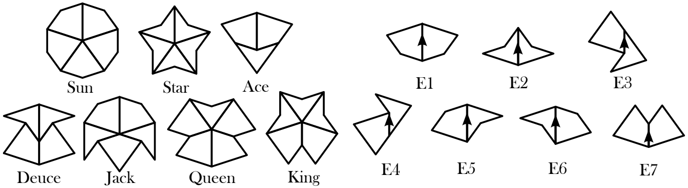
Define a *k-cell type* to be an equivalence class of k-cell of the tiling, decorated with the local patch information of all tiles intersecting the (open) cell, taken up to translation/rigid motion. We shall call a k-cell type a *vertex, edge, or face type* for $k = 0, 1, 2$, resp.



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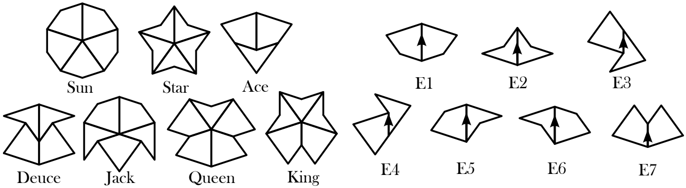
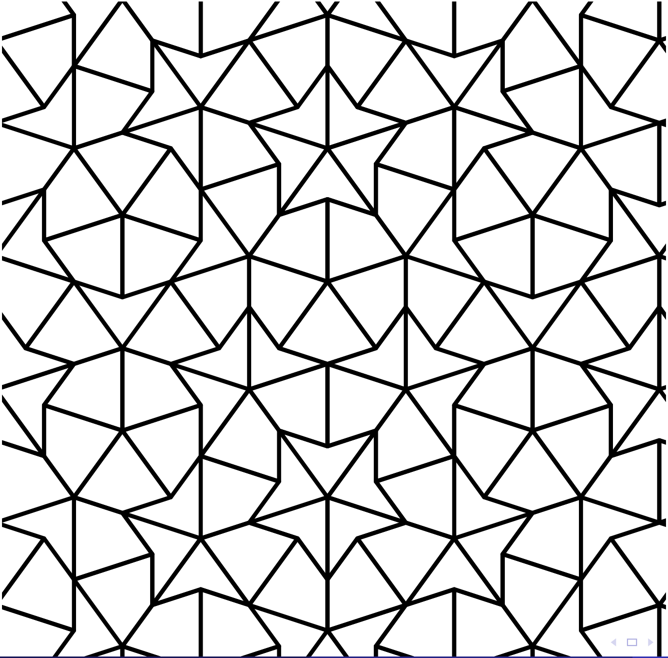
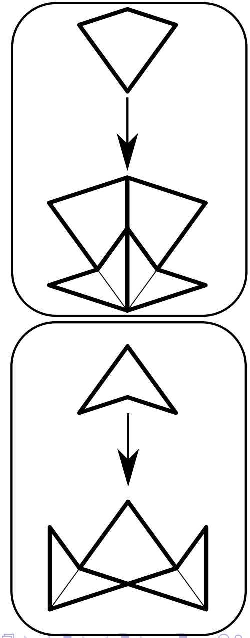
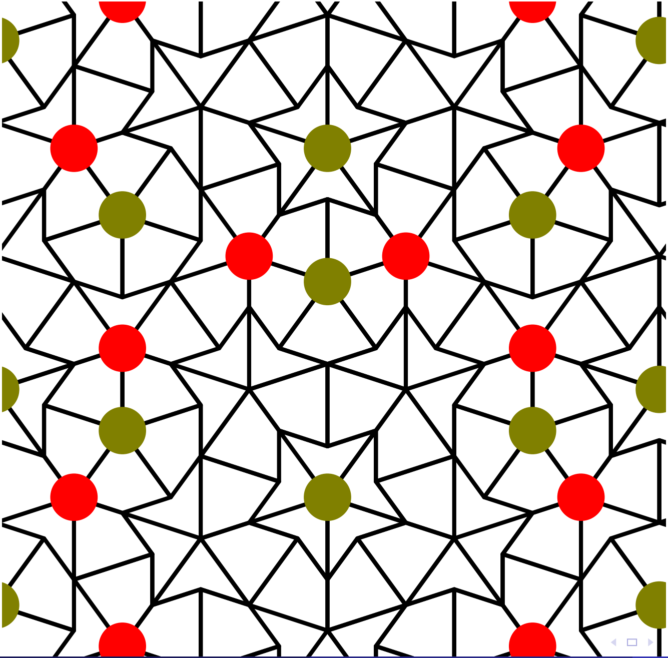
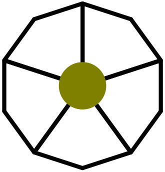


An assignment of coefficients to the k-cell types defines a $\mathcal{T}_i$E k-chain in the obvious way, and the cellular boundary map restricts to this complex; denote the resulting homology groups, the *approximant homology groups*, by $H_\bullet^{\mathcal{A}}$.

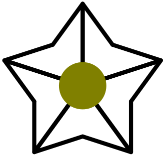




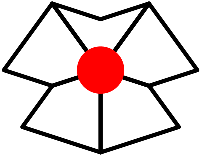




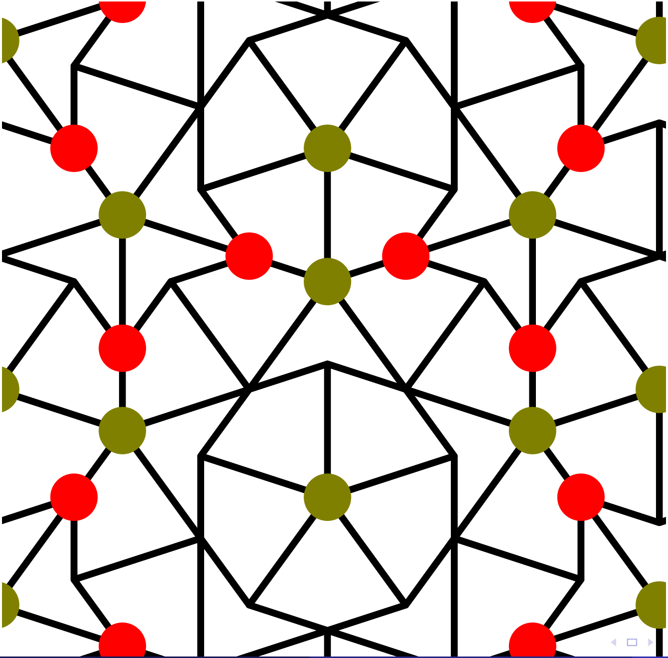
+1 Sun



+1 Star



-1 Queen



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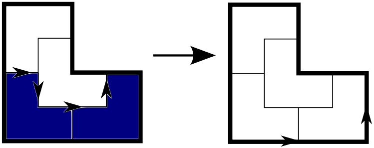


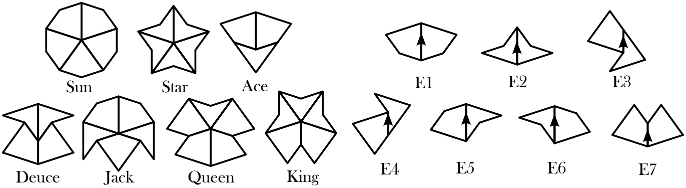
This defines the *substitution homomorphism* $\omega_\bullet : H_\bullet^{\mathcal{A}} \to H_\bullet^{\mathcal{A}}$.

Theorem (W)

There is an isomorphism

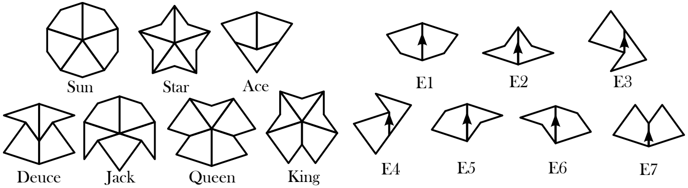
$$H_\bullet^{\mathcal{T}_i}(\mathbb{R}^d) \cong \varinjlim_{\omega} H_\bullet^{\mathcal{A}}.$$

Approximant Homology Groups



$$0 \xleftarrow[\partial_0]{} \mathbb{Z}^7 \xleftarrow[\partial_1]{} \mathbb{Z}^7 \xleftarrow[\partial_2]{} \mathbb{Z}^2 \leftarrow 0$$

Approximant Homology Groups



$$\partial_1 : \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & -2 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

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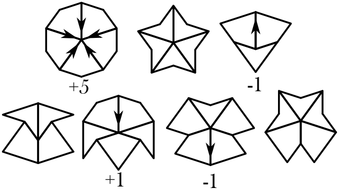

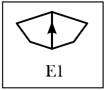

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We calculate:

$$H_0^{\mathcal{A}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_5$$
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The substitution homomorphisms ω_k are isomorphisms on homology so the above groups are isomorphic to the pattern-equivariant homology groups $H_\bullet^{\mathcal{T}_0}(\mathbb{R}^d)$ of the Kite and Dart Tiling.

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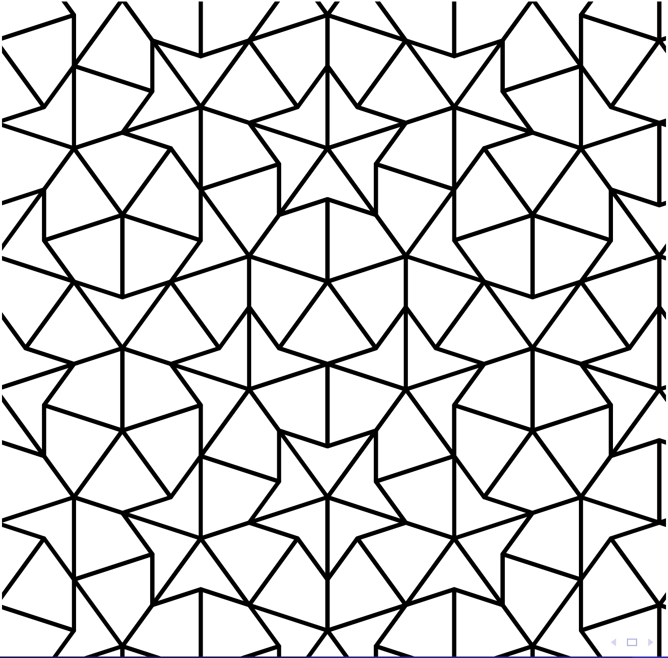
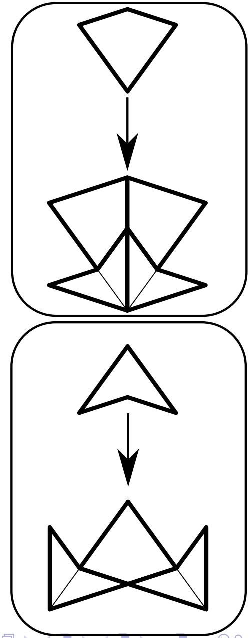
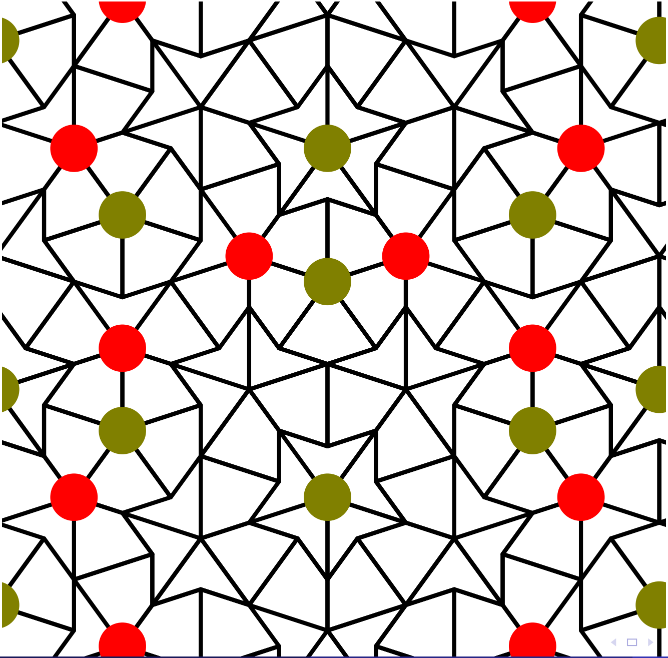
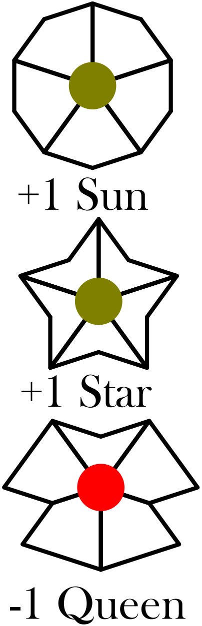
We don't get Poincaré duality in degree zero:

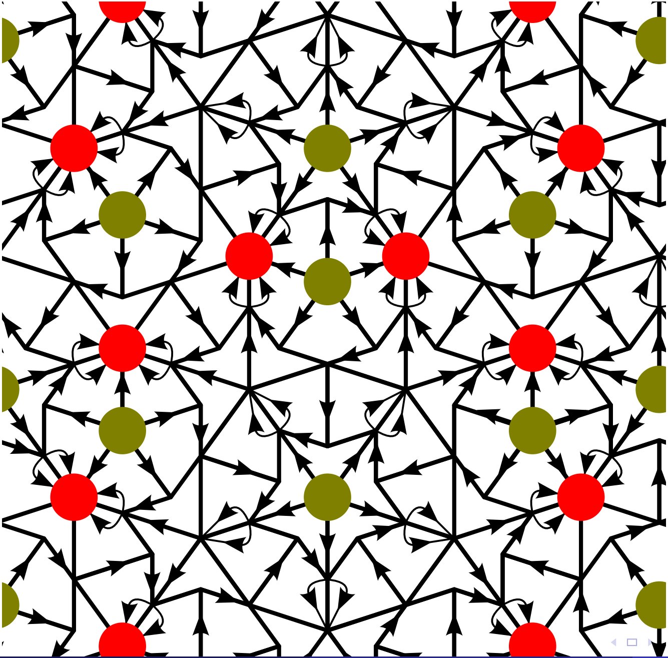
$$\check{H}^2(\Omega_0) \cong \mathbb{Z}^2$$

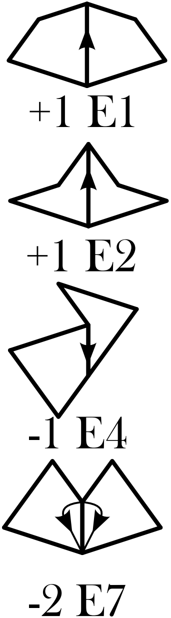


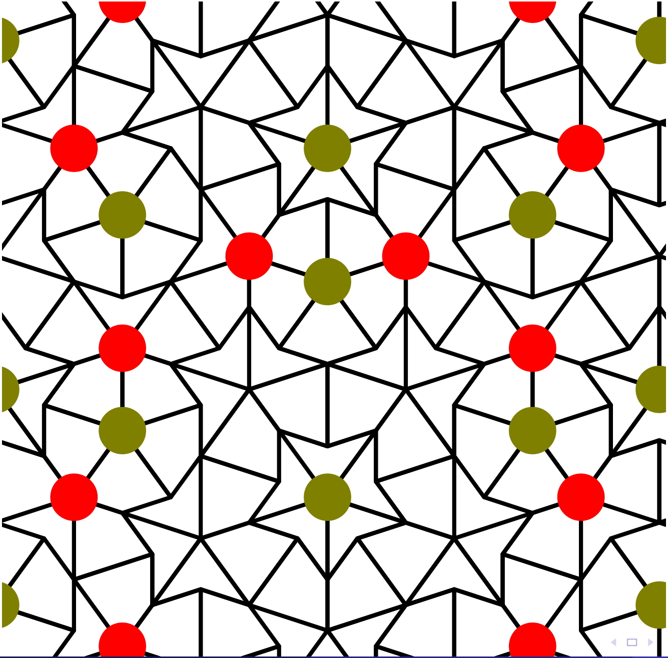


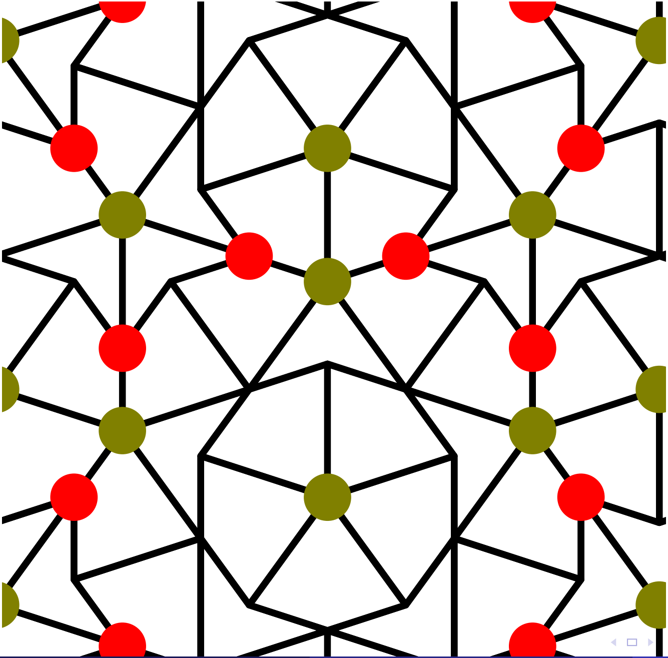


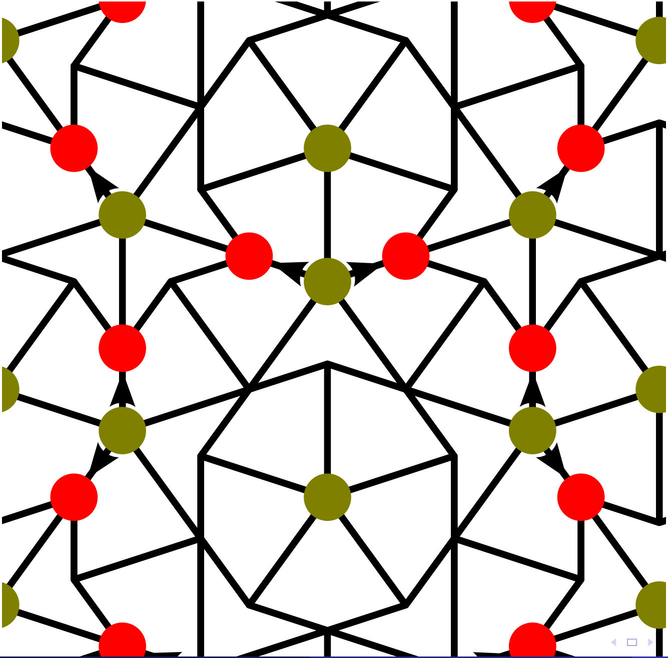

+1 Sun
+1 Star
-1 Queen

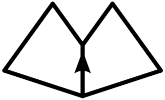



+1 E1
+1 E2
-1 E4
-2 E7









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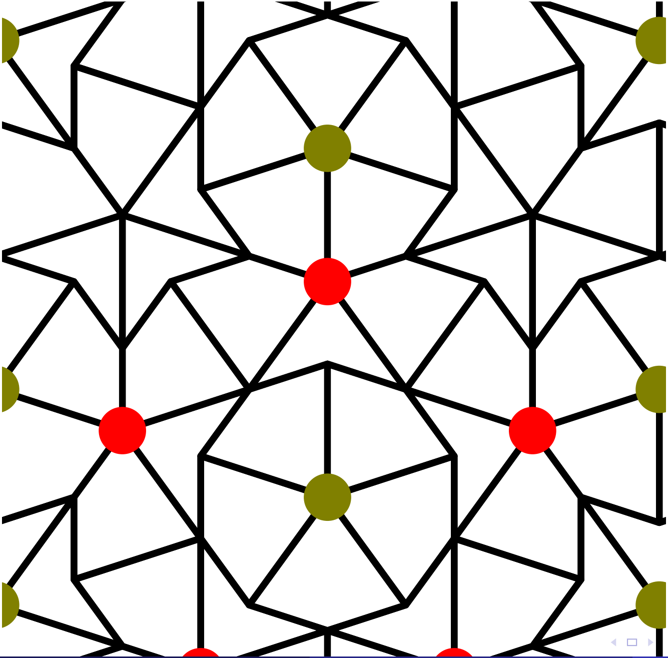

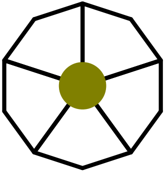

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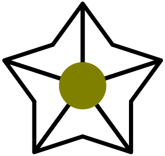

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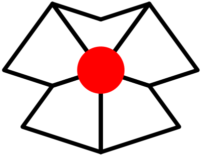
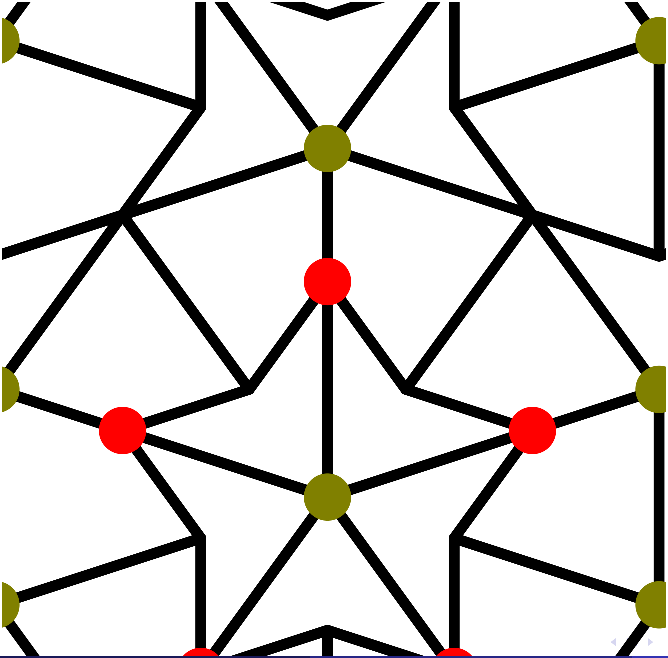
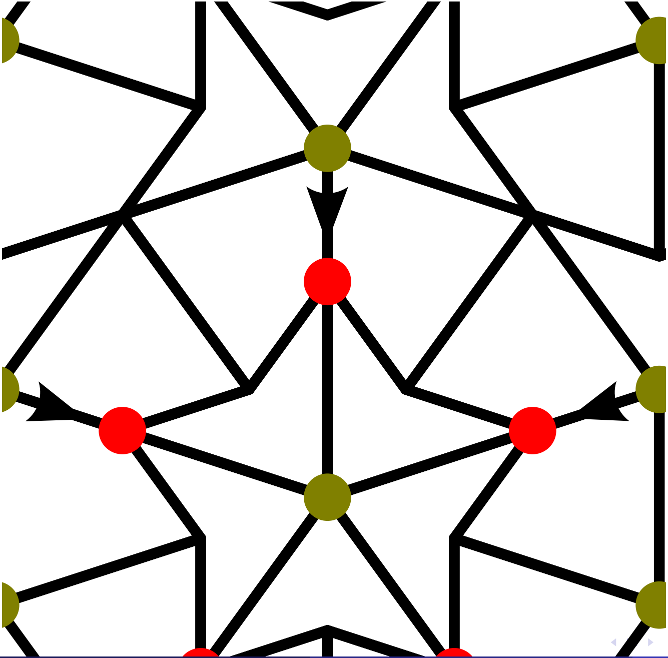
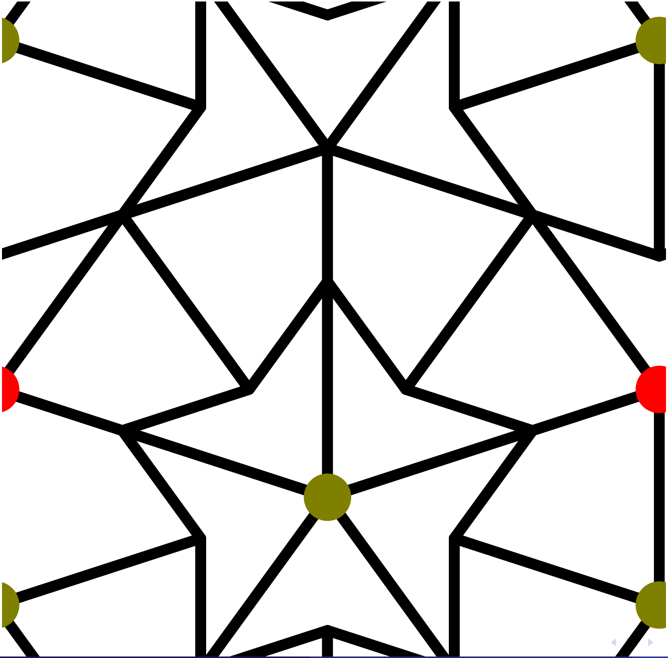
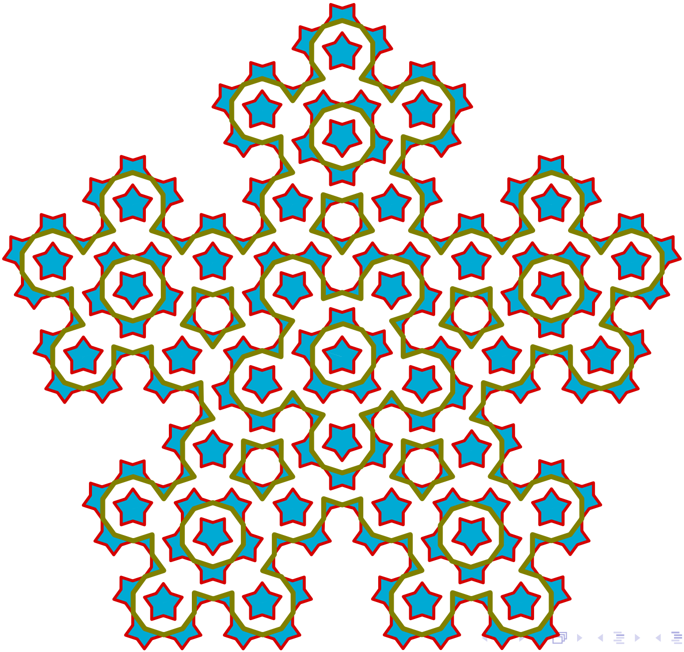


-1 Queen









Theorem (W)

Let T be an FLC (wrt rigid motions) tiling of $\mathbb{R}^2$. There is a spectral sequence with E_2 page

$$\begin{array}{c|ccc}
1 & H_0^{\mathcal{T}_0}(\mathbb{R}^2) & \check{H}^1(\Omega_0) & \check{H}^0(\Omega_0) \\
0 & \check{H}^2(\Omega_0) & \check{H}^1(\Omega_0) & \check{H}^0(\Omega_0) \\
\hline
 & 0 & 1 & 2
\end{array}$$

which converges to $H_k^{\mathcal{T}_{rot}}(E^+(2)) \cong \check{H}^{d-k}(\Omega_{rot})$.
Moreover, there is a simple construction of the image of the generator of $d_2 : \mathbb{Z} \to H_0^{\mathcal{T}_0}(\mathbb{R}^2)$ based on the k-cell types of T:

1) Pick a choice of vector for each face type:

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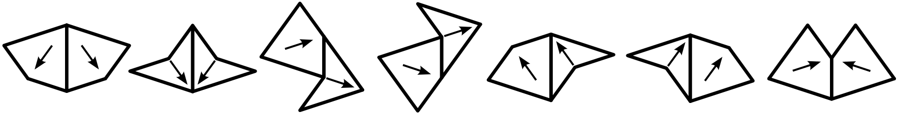
2) For each edge type, choose an ordering of the two face types and a rotation taking the vector of the first to the second:



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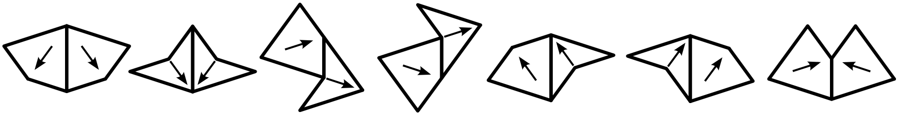
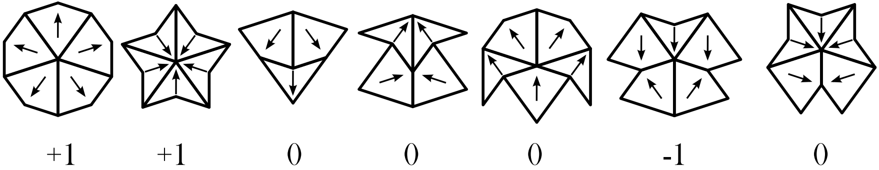
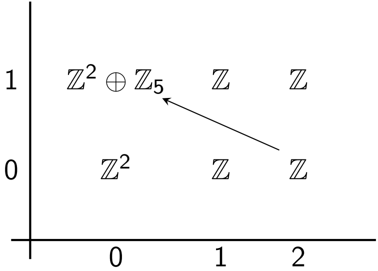


3) Calculate the winding number at each vertex type.



1	$H_0^{\mathcal{T}_0}(\mathbb{R}^2)$	$\check{H}^1(\Omega_0)$	$\check{H}^0(\Omega_0)$
0	$\check{H}^2(\Omega_0)$	$\check{H}^1(\Omega_0)$	$\check{H}^0(\Omega_0)$
	0	1	2

1	$\mathbb{Z}^2 \oplus \mathbb{Z}_5$	$\mathbb{Z}$	$\mathbb{Z}$
0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$
	0	1	2


1
$\mathbb{Z}^2 \oplus \mathbb{Z}_5$
$\mathbb{Z}$
$\mathbb{Z}$
0
$\mathbb{Z}^2$
$\mathbb{Z}$
$\mathbb{Z}$
0
1
2

1	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$
0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$
	0	1	2

So $\check{H}^k(\Omega_{\text{rot}}) \cong \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^2$ for $k = 0, 1, 2, 3,$ respectively.

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- The proof of the above spectral sequence is rather geometric and has no obvious generalisation to higher dimensions. What is the best way to generalise this method to higher dimensions (in a way which is *computable*)?

Thanks for listening!