Bak–Sneppen Type Models and Rank-Driven Processes

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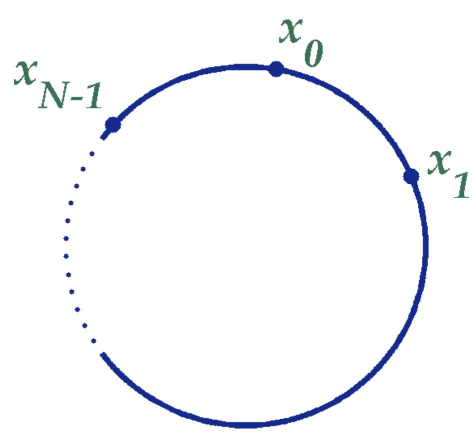
Outline

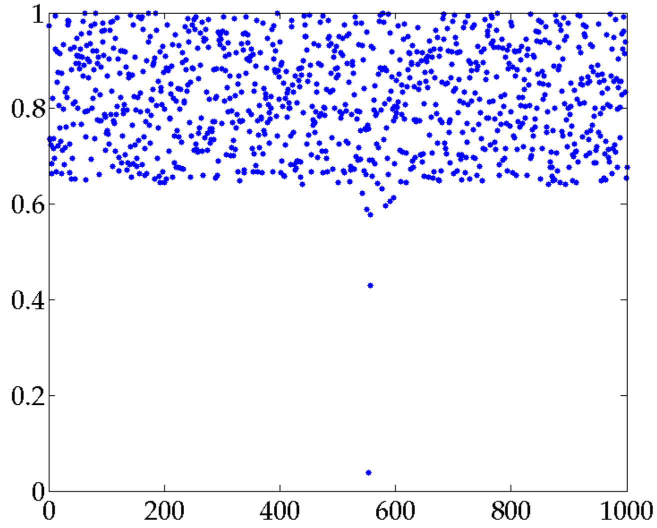
- Bak–Sneppen model and variants
- A measure of rank
- Rank driven processes
- Comparing the models
- Analysis of RDPs
- RDPs vs BS as extinction models

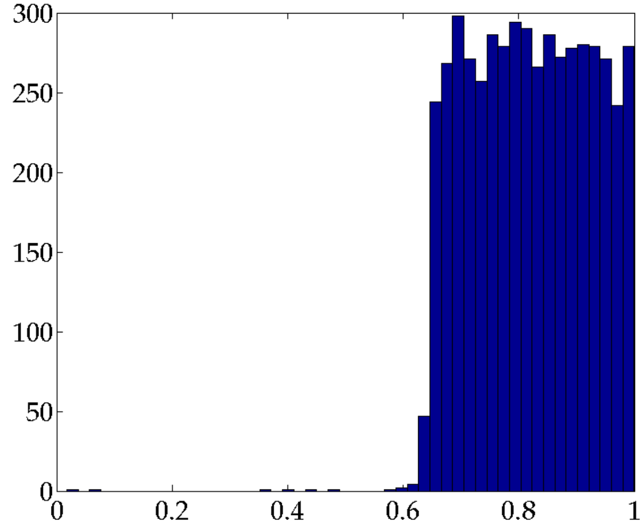
The Bak–Sneppen Model (1993)



- Model of extinction in species.
- N sites connected in a ring each allocated fitness in $U[0,1]$.
- Iteratively update fitnesses.
- At each step reallocate fitness of site with minimum fitness and its two neighbours.
- Cited around 900 times but few rigorous results.

Bak–Sneppen Dynamics





- Marginal distribution of fitness evolves to $U[s^*, 1]$.
- As $N \longrightarrow \infty$, $s^* \approx 0.667$.
- Avalanche has length l if min fitness stays below s for l steps.
- Distibution of $l \approx Cl^{-\tau}$ as $s \longrightarrow s^*$ ($\tau \approx 1.1$).
- Proven: steady-state distribution non-trivial.

Variants

- Number of neighbours updated.
- Topology: d-dimesional lattice, small world graph.
- Varying number of species.
- Different number of fitnesses.
- Anisotropic BS (aBS): only update right-hand neighbour. $s^* \approx 0.724$.
- Mean field: update smallest and $K-1$ randomly chosen sites. $s^*_K = 0.5$, $\tau_K = 1.5$ for $K = 2$.
- BS can be viewed as a Markov chain on $[0,1]^N$.
- Can we find a Markov process on a countable state space with equivalent dynamics?

Rank of Updated Fitness

- Given a set $\{x_k\}_{k=0}^{N-1}$ we order then to give $\{x_{(k)}\}_{k=0}^{N-1}$ where

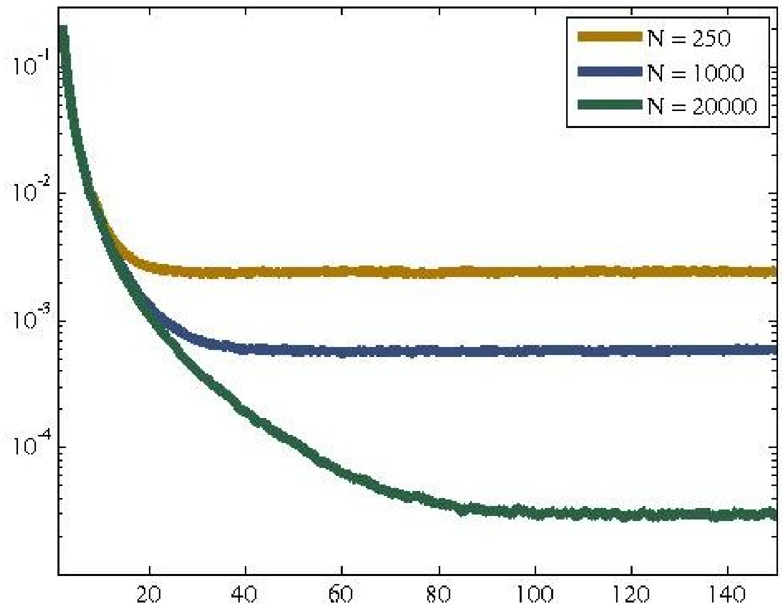
$$0 \leq x_{(0)} \leq \cdots \leq x_{(N-1)} \leq 1.$$

- $x_{(k)}$ is kth ranked element.
- Define $f_N(k)$ to be equilibrium probability that the right nearest neighbour of the smallest element is k-th ranked.
- Let $P(k, M)$ be the number of times the k-th ranked element is the right neighbour of the smallest element in M iterations of BS.

$$f_N(k) = \lim_{M \to \infty} \frac{1}{M} P(k, M).$$

- Left/aBS neighbour distribution similarly defined.

$f_N(k)$ for aBS

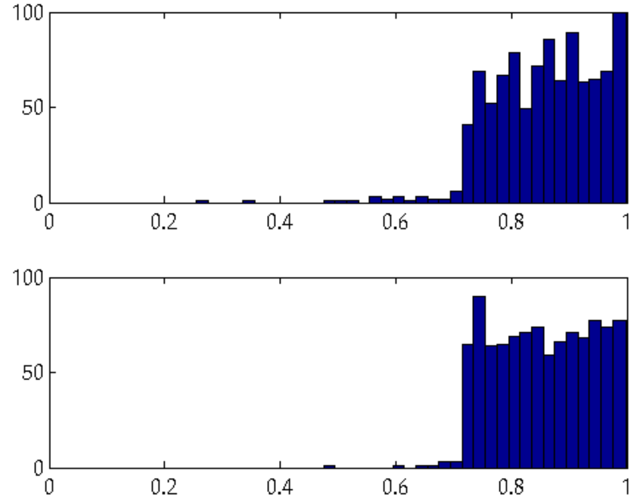


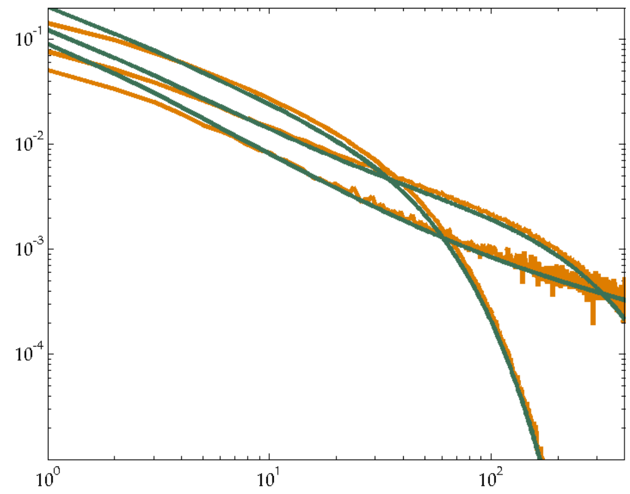
- $f_N(1) \approx \mathbf{0.21}$.
- f_N rapidly decays to a constant value.

Rank Driven Processes

- At each step, $K-1$ of the x_k-values are selected according to rank by sampling from $\{1,2,\ldots,N-1\}$ using f_N.
- The sample from $\{1,2,\ldots,N\}$ specifies the $x_{(k)}$ that are chosen.
- The chosen x_k-values are replaced by new (independent) random numbers, e.g. from $U[0,1]$.
- Mean field: f_N uniform.

RDP v aBS





- Dynamics of RDP are remarkably similar to aBS.
- Topology of aBS not key.
- Similar results for BS, too.

Analysis of RDPs

$$\alpha = \lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=1}^{n} f_N(k)$$

- $\alpha \in [0,1]$ measures the "atomicity" of f_N as $N \longrightarrow \infty$.
- In mean field ($K = 2$), $\alpha = 0$.
- If we always replace $x_{(0)}$ and $x_{(1)}$, $\alpha = 1$.

⋆ RDPs ($K = 2$) have a threshold at $s^* = (1 + \alpha)/2$.

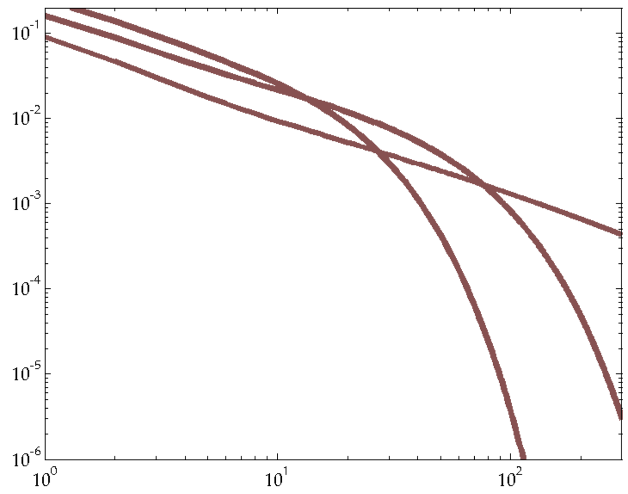
⋆ Limiting marginal is $U[s^*, 1]$ if f_N is "eventually uniform".

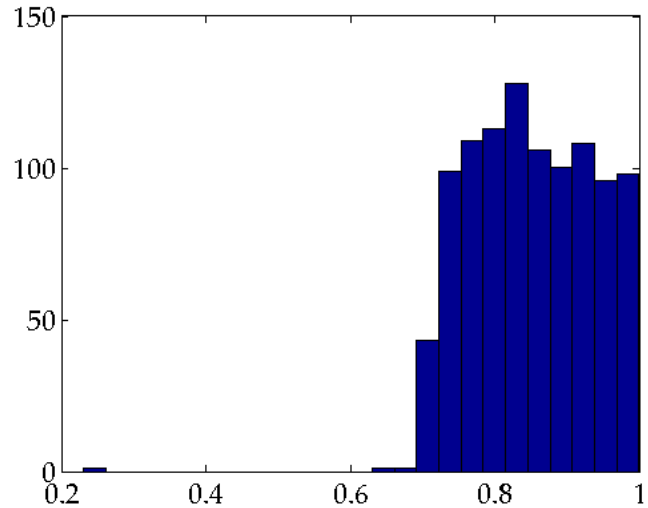
- f_N for aBS gives $\alpha \approx 0.445$.
- Conjecture: $s_K^* = (1 + (K - 1)\alpha)/K$.
- Aim to link α to τ.

Why The Close Connection?

- The exact relationship of aBS and RDP remains to be characterized rigorously.
- We can project aBS onto order statistics.
- Similarity suggests this projection is the same as RDP.
- Weaker asymptotic connection of key statistics may be sufficient explanation.
- A formal tie allows us to extend RDP results to aBS.

SOC in RDPs





- $f_N(k) = \begin{cases} 0.2, & k = 1, 2, \\ \frac{0.6}{N-2}, & k > 2. \end{cases}$
- $s^* = 0.7$.
- $\tau \in [1, 1.6]$.

RDPs as Extinction Models

- Motivation for RDPs was to analyse BS.
- RDPs can be used as extinction models in there own right.
- Local topological connections replaced by global statistical connections without compromising dynamics.
- Possibility of empirical input.
- Need for external agent can be removed.
- Threshold only emerges if our extinction model is guided by both random and selective evolutionary forces.